IMC 1994

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– Day 1

1 a) Let A be a $n \times n$, $n \geq 2$, symmetric, invertible matrix with real positive elements. Show that $z_n \leq n^2 - 2n$, where z_n is the number of zero elements in A^{-1}.

b) How many zero elements are there in the inverse of the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 1 & 2 & \dots & \ddots \end{pmatrix}$$

2 Let $f \in C^1(a,b)$, $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a,b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

3 Given a set S of $2n-1$, $n \in \mathbb{N}$, different irrational numbers. Prove that there are n different elements $x_1, x_2, \dots, x_n \in S$ such that for all non-negative rational numbers $a_1, a_2, \dots, a_n$ with $a_1 + a_2 + \dots + a_n > 0$ we have that $a_1x_1 + a_2x_2 + \dots + a_nx_n$ is an irrational number.

4 Let $\alpha \in \mathbb{R} \backslash \{0\}$ and suppose that F and G are linear maps (operators) from $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying $F \circ G - G \circ F = \alpha F$.

a) Show that for all $k \in \mathbb{N}$ one has $F^k \circ G - G \circ F^k = \alpha k F^k$.

b) Show that there exists $k \geq 1$ such that $F^k = 0$.

5 a) Let $f \in C[0,b]$, $g \in C(\mathbb{R})$ and let g be periodic with period b. Prove that $\int_0^b f(x)g(nx)\,\mathrm{d}x$ has a limit as $n \to \infty$ and

$$\lim_{n \to \infty} \int_0^b f(x)g(nx)\,\mathrm{d}x = \frac{1}{b} \int_0^b f(x)\,\mathrm{d}x \cdot \int_0^b g(x)\,\mathrm{d}x$$

b) Find

$$\lim_{n \to \infty} \int_0^{\pi} \frac{\sin x}{1 + 3\cos^2 nx}\,\mathrm{d}x$$

6 Let $f \in C^2[0,N]$ and $|f'(x)| < 1$, $f''(x) > 0$ for every $x \in [0,N]$. Let $0 \le m_0 < m_1 < \dots < m_k \le N$ be integers such that $n_i = f(m_i)$ are also integers for $i = 0, 1, \dots, k$. Denote $b_i = n_i - n_{i-1}$ and $a_i = m_i - m_{i-1}$ for $i = 1, 2, \dots, k$.

a) Prove that

$$-1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \dots < \frac{b_k}{a_k} < 1$$

b) Prove that for every choice of $A > 1$ there are no more than N/A indices j such that $a_j > A$.

c) Prove that $k \le 3N^{2/3}$ (i.e. there are no more than $3N^{2/3}$ integer points on the curve $y = f(x)$, $x \in [0,N]$).

– Day 2

1 Let $f \in C^1[a,b]$, $f(a) = 0$ and suppose that $\lambda \in \mathbb{R}$, $\lambda > 0$, is such that

$$|f'(x)| \le \lambda |f(x)|$$

for all $x \in [a,b]$. Is it true that $f(x) = 0$ for all $x \in [a,b]$?

2 Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$.

a) Prove that f attains its minimum and its maximum.

b) Determine all points (x,y) such that $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$ and determine for which of them f has global or local minimum or maximum.

3 Let f be a real-valued function with $n+1$ derivatives at each point of $\mathbb{R}$. Show that for each pair of real numbers a, b, $a < b$, such that

$$\ln\left(\frac{f(b) + f'(b) + \dots + f^{(n)}(b)}{f(a) + f'(a) + \dots + f^{(n)}(a)}\right) = b - a$$

there is a number c in the open interval (a,b) for which

$$f^{(n+1)}(c) = f(c)$$

4 Let A be a $n \times n$ diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \dots (x - c_k)^{d_k}$$

where $c_1, c_2, \dots, c_k$ are distinct (which means that c_1 appears d_1 times on the diagonal, c_2 appears d_2 times on the diagonal, etc. and $d_1 + d_2 + \dots + d_k = n$).

Let V be the space of all $n \times n$ matrices B such that $AB = BA$. Prove that the dimension of V is

$$d_1^2 + d_2^2 + \cdots + d_k^2$$

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5 **problem 5.**

Let $x_1, x_2, \ldots, x_k$ be vectors of m-dimensional Euclidean space, such that $x_1 + x_2 + \ldots + x_k = 0$. Show that there exists a permutation π of the integers $\{1, 2, \ldots, k\}$ such that:

$$\left\| \sum_{i=1}^{n} x_{\pi(i)} \right\| \le \left(\sum_{i=1}^{k} \|x_i\|^2 \right)^{1/2}$$

for each $n = 1, 2, \ldots, k$. Note that $\|\cdot\|$ denotes the Euclidean norm.
(18 points).

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6 Find

$$\lim_{N \to \infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)}$$
