Junior Balkan MO 2013

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by Igor

1 Find all ordered pairs (a, b) of positive integers for which the numbers $\dfrac{a^3b-1}{a+1}$ and $\dfrac{b^3a+1}{b-1}$ are both positive integers.

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2 Let ABC be an acute-angled triangle with $AB < AC$ and let O be the centre of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear.

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3 Show that

$$\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right) \geq 16$$

for all positive real numbers a and b such that $ab \geq 1$.

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4 Let n be a positive integer. Two players, Alice and Bob, are playing the following game:
- Alice chooses n real numbers; not necessarily distinct.
- Alice writes all pairwise sums on a sheet of paper and gives it to Bob. (There are $\frac{n(n-1)}{2}$ such sums; not necessarily distinct.)
- Bob wins if he finds correctly the initial n numbers chosen by Alice with only one guess.
Can Bob be sure to win for the following cases?

a. $n = 5$
b. $n = 6$
c. $n = 8$

Justify your answer(s).

[For example, when $n = 4$, Alice may choose the numbers 1, 5, 7, 9, which have the same pairwise sums as the numbers 2, 4, 6, 10, and hence Bob cannot be sure to win.]
