Czech And Slovak Mathematical Olympiad, Round III, Category A 2002

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by Sayan

1 Solve the system

$$(4x)_5 + 7y = 14(2y)_5 - (3x)_7 = 74$$

in the domain of integers, where $(n)_k$ stands for the multiple of the number k closest to the number n.

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2 Consider an arbitrary equilateral triangle KLM, whose vertices K, L and M lie on the sides AB, BC and CD, respectively, of a given square $ABCD$. Find the locus of the midpoints of the sides KL of all such triangles KLM.

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3 Show that a given natural number A is the square of a natural number if and only if for any natural number n, at least one of the differences

$$(A+1)^2 - A, (A+2)^2 - A, (A+3)^2 - A, \cdots, (A+n)^2 - A$$

is divisible by n.

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4 Find all pairs of real numbers a, b for which the equation in the domain of the real numbers

$$\frac{ax^2 - 24x + b}{x^2 - 1} = x$$

has two solutions and the sum of them equals 12.

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5 A triangle KLM is given in the plane together with a point A lying on the half-line opposite to KL. Construct a rectangle $ABCD$ whose vertices B, C and D lie on the lines KM, KL and LM, respectively. (We allow the rectangle to be a square.)

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6 Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying for all $x, y \in \mathbb{R}^+$ the equality

$$f(xf(y)) = f(xy) + x$$
