Czech And Slovak Mathematical Olympiad, Round III, Category A 2007

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by littletush

1 A stone is placed in a square of a chessboard with n rows and n columns. We can alternately undertake two operations:
(a) move the stone to a square that shares a common side with the square in which it stands;
(b) move it to a square sharing only one common vertex with the square in which it stands.

In addition, we are required that the first step must be **(b)**. Find all integers n such that the stone can go through a certain path visiting every square exactly once.

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2 In a cyclic quadrilateral $ABCD$, let L and M be the incenters of ABC and BCD respectively. Let R be a point on the plane such that $LR\perp AC$ and $MR\perp BD$.Prove that triangle LMR is isosceles.

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3 Consider a function $f:\mathbb{N}\to\mathbb{N}$ such that for any two positive integers x,y, the equation $f(xf(y))=yf(x)$ holds. Find the smallest possible value of $f(2007)$.

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4 The set $M=\{1,2,\dots,2007\}$ has the following property: If n is an element of M, then all terms in the arithmetic progression with its first term n and common difference $n+1$, are in M. Does there exist an integer m such that all integers greater than m are elements of M?

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5 In an acute-angled triangle ABC ($AC\neq BC$), let D and E be points on BC and AC, respectively, such that the points A,B,D,E are concyclic and AD intersects BE at P. Knowing that $CP\perp AB$, prove that P is the orthocenter of triangle ABC.

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6 Find all pariwise distinct real numbers x,y,z such that $\left\{\frac{x-y}{y-z},\frac{y-z}{z-x},\frac{z-x}{x-y}\right\}=\{x,y,z\}$. (It means, those three fractions make a permutation of x,y, and z.)
