Czech And Slovak Mathematical Olympiad, Round III, Category A 2008

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Day 1

1 Find all pairs of real numbers (x, y) satisfying:

$$x+y^2=y^3,$$
$$y+x^2=x^3.$$

2 Two disjoint circles $W_1(S_1,r_1)$ and $W_2(S_2,r_2)$ are given in the plane. Point A is on circle W_1 and AB, AC touch the circle W_2 at B, C respectively. Find the loci of the incenter and orthocenter of triangle ABC.

3 Find all pairs of integers (a, b) such that $a^2+ab+1 \mid b^2+ab+a+b-1$.

Day 2

1 In decimal representation, we call an integer [i]k-carboxylic[/i] if and only if it can be represented as a sum of k distinct integers, all of them greater than 9, whose digits are the same. For instance, 2008 is [i]5-carboxylic[/i] because $2008=1111+666+99+88+44$. Find, with an example, the smallest integer k such that 8002 is [i]k-carboxylic[/i].

2 At one moment, a kid noticed that the end of the hour hand, the end of the minute hand and one of the twelve numbers (regarded as a point) of his watch formed an equilateral triangle. He also calculated that t hours would elapse for the next similar case. Suppose that the ratio of the lengths of the minute hand (whose length is equal to the distance from the center of the watch plate to any of the twelve numbers) and the hour hand is $k>1$. Find the maximal value of t.

3 Find the greatest value of p and the smallest value of q such that for any triangle in the plane, the inequality

$$p<\frac{a+m}{b+n}<q$$

holds, where a, b are it's two sides and m, n their corresponding medians.