Czech And Slovak Mathematical Olympiad, Round III, Category A 2009

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by littletush

1 Knowing that the numbers $p, 3p+2, 5p+4, 7p+6, 9p+8$, and $11p+10$ are all primes, prove that $6p+11$ is a composite number.

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2 Rectangle $ABCD$ is inscribed in circle O. Let the projections of a point P on minor arc CD onto AB, AC, BD be K, L, M, respectively. Prove that $\angle LKM = 45$if and only if $ABCD$ is a square.

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3 Find the least value of $x > 0$ such that for all positive real numbers a, b, c, d satisfying $abcd = 1$, the inequality $a^x + b^x + c^x + d^x \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ is true.

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4 A positive integer n is called *good* if and only if there exist exactly 4 positive integers k_1, k_2, k_3, k_4 such that $n + k_i | n + k_i^2$ $(1 \leq k \leq 4)$. Prove that:

-58 is *good*;

-$2p$ is *good* if and only if p and $2p+1$ are both primes ($p > 2$).

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5 At every vertex $A_k (1 \leq k \leq n)$ of a regular n-gon, k coins are placed. We can do the following operation: in each step, one can choose two arbitrarily coins and move them to their adjacent vertices respectively, one clockwise and one anticlockwise. Find all positive integers n such that after a finite number of operations, we can reach the following configuration: there are $n+1-k$ coins at vertex A_k for all $1 \leq k \leq n$.

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6 Given two fixed points O and G in the plane. Find the locus of the vertices of triangles whose circumcenters and centroids are O and G respectively.
