Czech And Slovak Mathematical Olympiad, Round III, Category A 2011

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by Shu

Day 1

1 In a certain triangle ABC, there are points K and M on sides AB and AC, respectively, such that if L is the intersection of MB and KC, then both $AKLM$ and $KBCM$ are cyclic quadrilaterals with the same size circumcircles. Find the measures of the interior angles of triangle ABC.

2 Find all triples (p, q, r) of prime numbers for which

$$(p+1)(q+2)(r+3) = 4pqr.$$

3 Suppose that x, y, z are real numbers satisfying

$$x+y+z=12, \qquad \text{and} \qquad x^2+y^2+z^2=54.$$

Prove that:(a) Each of the numbers xy, yz, zx is at least 9, but at most 25.
(b) One of the numbers x, y, z is at most 3, and another one is at least 5.

Day 2

4 Consider a quadratic polynomial ax^2+bx+c with real coefficients satisfying $a \geq 2$, $b \geq 2$, $c \geq 2$. Adam and Boris play the following game. They alternately take turns with Adam first. On Adams turn, he can choose one of the polynomials coefficients and replace it with the sum of the other two coefficients. On Boriss turn, he can choose one of the polynomials coefficients and replace it with the product of the other two coefficients. The winner is the player who first produces a polynomial with two distinct real roots. Depending on the values of a, b and c, determine who has a winning strategy.

5 In acute triangle ABC, which is not equilateral, let P denote the foot of the altitude from C to side AB; let H denote the orthocenter; let O denote the circumcenter; let D denote the intersection of line CO with AB; and let E denote the midpoint of CD. Prove that line EP passes through the midpoint of OH.

6 Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $x, y \in \mathbb{R}^+$, we have

$$f(x)f(y) = f(y)f\Big(xf(y)\Big) + \frac{1}{xy}.$$