China Team Selection Test 2017

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– **TST 1**

– **Day 1**

1 Find out the maximum value of the numbers of edges of a solid regular octahedron that we can see from a point out of the regular octahedron.(We define we can see an edge AB of the regular octahedron from point P outside if and only if the intersection of non degenerate triangle PAB and the solid regular octahedron is exactly edge AB.

2 Let $x>1$,n be positive integer. Prove that

$$\sum_{k=1}^{n}\frac{\{kx\}}{[kx]}<\sum_{k=1}^{n}\frac{1}{2k-1}$$

Where $[kx]$ be the integer part of kx ,$\{kx\}$ be the decimal part of kx.

3 Suppose $S=\{1,2,3,...,2017\}$,for every subset A of S,define a real number $f(A)\geq 0$ such that: (1) For any $A,B\subset S$,$f(A\cup B)+f(A\cap B)\leq f(A)+f(B)$; (2) For any $A\subset B\subset S$, $f(A)\leq f(B)$; (3) For any $k,j\in S$,

$$f(\{1,2,\ldots,k+1\})\geq f(\{1,2,\ldots,k\}\cup \{j\});$$

(4) For the empty set $\varnothing$, $f(\varnothing)=0$.
Confirm that for any three-element subset T of S,the inequality

$$f(T)\leq \frac{27}{19}f(\{1,2,3\})$$

holds.

– **Day 2**

4 Find out all the integer pairs (m,n) such that there exist two monic polynomials $P(x)$ and $Q(x)$,with $\deg{P}=m$ and $\deg{Q}=n$,satisfy that

$$P(Q(t))\neq Q(P(t))$$

holds for any real number t.

5 In the non-isosceles triangle ABC,D is the midpoint of side BC,E is the midpoint of side CA,F is the midpoint of side AB.The line(different from line BC) that is tangent to the inscribed circle of triangle ABC and passing through point D intersect line EF at X.Define Y, Z similarly.Prove that X, Y, Z are collinear.

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6 For a given positive integer n and prime number p, find the minimum value of positive integer m that satisfies the following property: for any polynomial

$$f(x) = (x + a_1)(x + a_2) \dots (x + a_n)$$

($a_1, a_2, \dots, a_n$ are positive integers), and for any non-negative integer k, there exists a non-negative integer k' such that

$$v_p(f(k)) < v_p(f(k')) \leq v_p(f(k)) + m.$$

Note: for non-zero integer N,$v_p(N)$ is the largest non-zero integer t that satisfies $p^t \mid N$.

– **TST 2**

– **Day 1**

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1 Let n be a positive integer. Let D_n be the set of all divisors of n and let $f(n)$ denote the smallest natural m such that the elements of D_n are pairwise distinct in mod m. Show that there exists a natural N such that for all $n \geq N$, one has $f(n) \leq n^{0.01}$.

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2 2017 engineers attend a conference. Any two engineers if they converse, converse with each other in either Chinese or English. No two engineers converse with each other more than once. It is known that within any four engineers, there was an even number of conversations and furthermore within this even number of conversations:

i) At least one conversation is in Chinese.
ii) Either no conversations are in English or the number of English conversations is at least that of Chinese conversations.

Show that there exists 673 engineers such that any two of them conversed with each other in Chinese.

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3 Let $ABCD$ be a quadrilateral and let l be a line. Let l intersect the lines AB, CD, BC, DA, AC, BD at points X, X', Y, Y', Z, Z' respectively. Given that these six points on l are in the order X, Y, Z, X', Y', Z', show that the circles with diameter XX', YY', ZZ' are coaxal.

– **Day 2**

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4 An integer $n > 1$ is given . Find the smallest positive number m satisfying the following conditions for any set $\{a, b\} \subset \{1, 2, \cdots, 2n-1\}$,there are non-negative integers x, y (not all zero) such that $2n|ax + by$ and $x + y \leq m$.

5 Let $\varphi(x)$ be a cubic polynomial with integer coefficients. Given that $\varphi(x)$ has have 3 distinct real roots u, v, w and u, v, w are not rational number. there are integers a, b, c such that $u = av^2 + bv + c$. Prove that $b^2 - 2b - 4ac - 7$ is a square number .

6 Let M be a subset of $\mathbb{R}$ such that the following conditions are satisfied:

a) For any $x \in M, n \in \mathbb{Z}$, one has that $x + n \in \mathbb{M}$.
b) For any $x \in M$, one has that $-x \in M$.
c) Both M and $\mathbb{R}\ \ M$ contain an interval of length larger than 0.

For any real x, let $M(x) = \{n \in \mathbb{Z}^+ | nx \in M\}$. Show that if α, β are reals such that $M(\alpha) = M(\beta)$, then we must have one of $\alpha + \beta$ and $\alpha - \beta$ to be rational.

– **TST 3**

– **Day 1**

1 Let $n \geq 4$ be a natural and let $x_1, \ldots, x_n$ be non-negative reals such that $x_1 + \cdots + x_n = 1$. Determine the maximum value of $x_1x_2x_3 + x_2x_3x_4 + \cdots + x_nx_1x_2$.

2 Let $ABCD$ be a non-cyclic convex quadrilateral. The feet of perpendiculars from A to BC, BD, CD are P, Q, R respectively, where P, Q lie on segments BC, BD and R lies on CD extended. The feet of perpendiculars from D to AC, BC, AB are X, Y, Z respectively, where X, Y lie on segments AC, BC and Z lies on BA extended. Let the orthocenter of $\triangle ABD$ be H. Prove that the common chord of circumcircles of $\triangle PQR$ and $\triangle XYZ$ bisects BH.

3 Let X be a set of 100 elements. Find the smallest possible n satisfying the following condition: Given a sequence of n subsets of X, $A_1, A_2, \ldots, A_n$, there exists $1 \leq i < j < k \leq n$ such that

$$A_i \subseteq A_j \subseteq A_k \text{ or } A_i \supseteq A_j \supseteq A_k.$$

– **Day 2**

4 Show that there exists a degree 58 monic polynomial

$$P(x) = x^{58} + a_1x^{57} + \cdots + a_{58}$$

such that $P(x)$ has exactly 29 positive real roots and 29 negative real roots and that $\log_{2017} |a_i|$ is a positive integer for all $1 \leq i \leq 58$.

5 Show that there exists a positive real C such that for any naturals H, N satisfying $H \geq 3, N \geq e^{CH}$, for any subset of $\{1, 2, \ldots, N\}$ with size $\lceil \frac{CHN}{\ln N} \rceil$, one can find H naturals in it such that the greatest common divisor of any two elements is the greatest common divisor of all H elements.

6 Every cell of a 2017×2017 grid is colored either black or white, such that every cell has at least one side in common with another cell of the same color. Let V_1 be the set of all black cells, V_2 be the set of all white cells. For set $V_i(i=1,2)$, if two cells share a common side, draw an edge with the centers of the two cells as endpoints, obtaining graphs G_i. If both G_1 and G_2 are connected paths (no cycles, no splits), prove that the center of the grid is one of the endpoints of G_1 or G_2.

– **TST 4**

– **Day 1**

1 Prove that :

$$\sum_{k=0}^{58} C_{2017+k}^{58-k} C_{2075-k}^{k} = \sum_{p=0}^{29} C_{4091-2p}^{58-2p}$$

2 In ΔABC,the excircle of A is tangent to segment BC,line AB and AC at E, D, F respectively.EZ is the diameter of the circle.B_1 and C_1 are on DF, and $BB_1\perp BC$,$CC_1\perp BC$.Line ZB_1, ZC_1 intersect BC at X, Y respectively.Line EZ and line DF intersect at H,ZK is perpendicular to FD at K.If H is the orthocenter of ΔXYZ,prove that:H, K, X, Y are concyclic.

3 Find the numbers of ordered array $(x_1, ..., x_{100})$ that satisfies the following conditions:
$(i)x_1, ..., x_{100} \in \{1, 2, .., 2017\}$;
$(ii)2017|x_1 + ... + x_{100}$;
$(iii)2017|x_1^2 + ... + x_{100}^2$.

– **Day 2**

4 Given integer $d > 1, m$,prove that there exists integer $k > l > 0$, such that

$$(2^{2^k} + d, 2^{2^l} + d) > m.$$

5 Given integer $m \geq 2$,$x_1, ..., x_m$ are non-negative real numbers,prove that:

$$(m-1)^{m-1}(x_1^m + ... + x_m^m) \geq (x_1 + ... + x_m)^m - m^m x_1 ... x_m$$

and please find out when the equality holds.

6 A plane has no vertex of a regular dodecahedron on it,try to find out how many edges at most may the plane intersect the regular dodecahedron?

– **TST 5**

1 Given $n \geq 3$. consider a sequence $a_1, a_2, ..., a_n$, if (a_i, a_j, a_k) with i+k=2j (i¡j¡k) and $a_i + a_k \neq 2a_j$, we call such a triple a $NOT - AP$ triple. If a sequence has at least one $NOT - AP$ triple, find the least possible number of the $NOT - AP$ triple it contains.

2 Find the least positive number m such that for any polynimial f(x) with real coefficients, there is a polynimial g(x) with real coefficients (degree not greater than m) such that there exist 2017 distinct number $a_1, a_2, ..., a_{2017}$ such that $g(a_i) = f(a_{i+1})$ for i=1,2,...,2017 where indices taken modulo 2017.

3 For a rational point (x,y), if xy is an integer that divided by 2 but not 3, color (x,y) red, if xy is an integer that divided by 3 but not 2, color (x,y) blue. Determine whether there is a line segment in the plane such that it contains exactly 2017 blue points and 58 red points.

4 Given a circle with radius 1 and 2 points C, D given on it. Given a constant l with $0 < l \leq 2$. Moving chord of the circle AB=l and ABCD is a non-degenerated convex quadrilateral. AC and BD intersects at P. Find the loci of the circumcenters of triangles ABP and BCP.

5 A(x,y), B(x,y), and C(x,y) are three homogeneous real-coefficient polynomials of x and y with degree 2, 3, and 4 respectively. we know that there is a real-coefficient polinimial R(x,y) such that $B(x,y)^2 - 4A(x,y)C(x,y) = -R(x,y)^2$. Proof that there exist 2 polynomials F(x,y,z) and G(x,y,z) such that $F(x,y,z)^2 + G(x,y,z)^2 = A(x,y)z^2 + B(x,y)z + C(x,y)$ if for any x, y, z real numbers $A(x,y)z^2 + B(x,y)z + C(x,y) \geq 0$

6 We call a graph with n vertices $k - flowing - chromatic$ if:
1. we can place a chess on each vertex and any two neighboring (connected by an edge) chesses have different colors.
2. we can choose a hamilton cycle $v_1, v_2, \cdots, v_n$, and move the chess on v_i to v_{i+1} with $i = 1, 2, \cdots, n$ and $v_{n+1} = v_1$, such that any two neighboring chess also have different colors.
3. after some action of step 2 we can make all the chess reach each of the n vertices.
Let T(G) denote the least number k such that G is k-flowing-chromatic.
If such k does not exist, denote T(G)=0.
denote $\chi(G)$ the chromatic number of G.
Find all the positive number m such that there is a graph G with $\chi(G) \leq m$ and $T(G) \geq 2^m$ without a cycle of length small than 2017.