Tournament Of Towns 2016

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—   [b]Spring 2016 - Junior A-level

1   All integers from 1 to one million are written on a tape in some arbitrary order. Then the tape is cut into pieces containing two consecutive digits each. Prove that these pieces contain all two-digit integers for sure, regardless of the initial order of integers.*(4 points)*
Alexey Tolpygo

2   Do there exist integers a and b such that :
(a) the equation $x^2 + ax + b = 0$ has no real roots, and the equation $\lfloor x^2 \rfloor + ax + b = 0$ has at least one real root?
(2 points)
(b) the equation $x^2 + 2ax + b$ = 0 has no real roots, and the equation $\lfloor x^2 \rfloor + 2ax + b = 0$ has at least one real root?
3 points
(By $\lfloor k \rfloor$ we denote the integer part of k, that is, the greatest integer not exceeding k.)
Alexandr Khrabrov

3   Given a square with side 10. Cut it into 100 congruent quadrilaterals such that each of them is inscribed into a circle with diameter $\sqrt{3}$. *(5 points)*

Ilya Bogdanov

4   A designer took a wooden cube $5 \times 5 \times 5$, divided each face into unit squares and painted each square black, white or red so that any two squares with a common side have different colours. What is the least possible number of black squares? (Squares with a common side may belong to the same face of the cube or to two different faces.)
(8 points)

Mikhail Evdokimov

5   Let p be a prime integer greater than 10^k. Pete took some multiple of p and inserted a $k-$digit integer A between two of its neighbouring digits. The resulting integer C was again a multiple of p. Pete inserted a $k-$digit integer B between two of neighbouring digits of C belonging to the inserted integer A, and the result was again a multiple of p. Prove that the integer B can be obtained from the integer A by a permutation of its digits.
(8 points)
Ilya Bogdanov

6 Q. An automatic cleaner of the disc shape has passed along a plain floor. For each point of its circular boundary there exists a straight line that has contained this point all the time. Is it necessarily true that the center of the disc stayed on some straight line all the time? (9 marks)

7 a.) There are $2n + 1$ ($n > 2$) batteries. We don't know which batteries are good and which are bad but we know that the number of good batteries is greater by 1 than the number of bad batteries. A lamp uses two batteries, and it works only if both of them are good. What is the least number of attempts sufficient to make the lamp work?

b.) The same problem but the total number of batteries is $2n$ ($n > 2$) and the numbers of good and bad batteries are equal.

Proposed by Alexander Shapovalov

– **Spring 2016 - Senior A-level**

1 Same as Junior A-level P1

2 Same as Junior A-level P3

3 Let M be the midpoint of the base AC of an isosceles $\triangle ABC$. Points E and F on the sides AB and BC respectively are chosen so that $AE \neq CF$ and $\angle FMC = \angle MEF = \alpha$. Determine $\angle AEM$. *(6 points)*

Maxim Prasolov

4 There are 64 towns in a country and some pairs of towns are connected by roads but we do not know these pairs. We may choose any pair of towns and find out whether they are connected or not. Our aim is to determine whether it is possible to travel from any town to any other by a sequence of roads. Prove that there is no algorithm which enables us to do so in less than 2016 questions.

(Proposed by Konstantin Knop)

5 On a blackboard, several polynomials of degree 37 are written, each of them has the leading co-efficient equal to 1. Initially all coefficients of each polynomial are non-negative. By one move it is allowed to erase any pair of polynomials f, g and replace it by another pair of polynomials f_1, g_1 of degree 37 with the leading coefficients equal to 1 such that either $f_1 + g_1 = f + g$ or $f_1 g_1 = fg$. Prove that it is impossible that after some move each polynomial on the blackboard has 37 distinct positive roots. *(8 points)*

Alexandr Kuznetsov

6 Recall that a palindrome is a word which is the same when we read it forward or backward.

(a) We have an infinite number of cards with words $\{abc, bca, cab\}$. A word is made from them in the following way. The initial word is an arbitrary card. At each step we obtain a new word either gluing a card (from the right or from the left) to the existing word or making a cut between any two of its letters and gluing a card between both parts. Is it possible to obtain a palindrome this way? *(4 points)*

(b) We have an infinite number of red cards with words $\{abc, bca, cab\}$ and of blue cards with words $\{cba, acb, bac\}$. A palindrome was formed from them in the same way as in part (a). Is it necessarily true that the number of red and blue cards used was equal? *(6 points)*

Alexandr Gribalko, Ivan Mitrofanov

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7   A spherical planet has the equator of length 1. On this planet, N circular roads of length 1 each are to be built and used for several trains each. The trains must have the same constant positive speed and never stop or collide. What is the greatest possible sum of lengths of all the trains? The trains are arcs of zero width with endpoints removed (so that if only endpoints of two arcs have coincided then it is not a collision). Solve the problem for :
(a) $N = 3$ *(4 points)*
(b) $N = 4$ *(6 points)*

Alexandr Berdnikov

–   **Oral Round**

1   On a blackboard the product $log_{()}[] \times \cdots \times log_{()}[]$ is written (there are 50 logarithms in the product). Donald has 100 cards: $[2], [3], \ldots, [51]$ and $(52), \ldots, (101)$. He is replacing each $()$ with some card of form (x) and each $[]$ with some card of form $[y]$. Find the difference between largest and smallest values Donald can achieve.

2   On plane there is fixed ray s with vertex A and a point P not on the line which contains s. We choose a random point K which lies on ray. Let N be a point on a ray outside AK such that $NK = 1$. Let M be a point such that $NM = 1, M \in PK$ and $M! = K$. Prove that all lines NM, provided by some point K, touch some fixed circle.

3   Rectangle $p * q$, where p, q are relatively coprime positive integers with $p < q$ is divided into squares $1 * 1$. Diagonal which goes from lowest left vertice to highest right cuts triangles from some squares. Find sum of perimeters of all such triangles.

4   30 masters and 30 juniors came onto tennis players meeting .Each master played with one master and 15 juniors while each junior played with one junior and 15 masters.Prove that one can find two masters and two juniors such that these masters played with each other ,juniors -with each other ,each of two masters played with at least one of two juniors and each of two juniors played with at least one of two masters.

5   In convex hexagonal pyramid 11 edges are equal to 1.Find all possible values of 12th edge.

6 N different numbers are written on blackboard and one of these numbers is equal to 0.One may take any polynomial such that each of its coefficients is equal to one of written numbers (there may be some equal coefficients) and write all its roots on blackboard.After some of these operations all integers between -2016 and 2016 were written on blackboard(and some other numbers maybe). Find the smallest possible value of N.

–   **Fall 2016 - Senior A-level**

1 100 children stand in a line each having 100 candies. In one move, one of them may take some of their candies and distribute them to a non-empty set of the remaining children. After what least number of moves can it happen that no two children have the same number of candies?

(N. Chernyatevya)

(Translated from here. (`http://sasja.shap.homedns.org/Turniry/TG/index.html`))

2   A natural number is written in each cell of an 8×8 board. It turned out that for any tiling of the board with dominoes, the sum of numbers in the cells of each domino is different. Can it happen that the largest number on the board is no greater than 32?

(N. Chernyatyev)

(Translated from here. (`http://sasja.shap.homedns.org/Turniry/TG/index.html`))

3   The quadrilateral $ABCD$ is inscribed in circle Ω with center O, not lying on either of the diagonals. Suppose that the circumcircle of triangle AOC passes through the midpoint of the diagonal BD. Prove that the circumcircle of triangle BOD passes through the midpoint of diagonal AC.

(A. Zaslavsky)

(Translated from here. (`http://sasja.shap.homedns.org/Turniry/TG/index.html`))

4   There are 2016 red and 2016 blue cards each having a number written on it. For some 64 distinct positive real numbers, it is known that the set of numbers on cards of a particular color happens to be the set of their pairwise sums and the other happens to be the set of their pairwise products. Can we necessarily determine which color corresponds to sum and which to product?

(B. Frenkin)

(Translated from here. (`http://sasja.shap.homedns.org/Turniry/TG/index.html`))

5   Is it possible to cut a square of side 1 into two parts and rearrange them so that one can cover a circle having diameter greater than 1?

(Note: any circle with diameter greater than 1 suffices)

(A. Shapovalov)

(Translated from here. (`http://sasja.shap.homedns.org/Turniry/TG/index.html`))

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6   Petya and Vasya play the following game. Petya conceives a polynomial $P(x)$ having integer coefficients. On each move, Vasya pays him a ruble, and calls an integer a of his choice, which has not yet been called by him. Petya has to reply with the number of distinct integer solutions of the equation $P(x) = a$. The game continues until Petya is forced to repeat an answer. What minimal amount of rubles must Vasya pay in order to win?

(Anant Mudgal)

(Translated from here. (`http://sasja.shap.homedns.org/Turniry/TG/index.html`))

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7   Several frogs are sitting on the real line at distinct integer points. In each move, one of them can take a 1-jump towards the right as long as they are still in on distinct points. We calculate the number of ways they can make N moves in this way for a positive integer N. Prove that if the jumps were all towards the left, we will still get the same number of ways.

(F. Petrov)

(Translated from here. (`http://sasja.shap.homedns.org/Turniry/TG/index.html`))