Israel National Olympiad 2017

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- In the right picture there is a square with four congruent circles inside it. Each circle is tangent to two others, and to two of the edges of the square. Evaluate the ratio between the blue part and white part of the square's area.
- In the left picture there is a regular hexagon with six congruent circles inside it. Each circle is tangent to two others, and to one of the edges on the hexagon in its midpoint. Evaluate the ratio between the blue part and white part of the hexagon's area.

`https://i.imgur.com/fAuxoc9.png`

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2 Denote by $P(n)$ the product of the digits of a positive integer n. For example, $P(1948) = 1 \cdot 9 \cdot 4 \cdot 8 = 288$.

- Evaluate the sum $P(1) + P(2) + \cdots + P(2017)$.
- Determine the maximum value of $\frac{P(n)}{n}$ where $2017 \le n \le 5777$.

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3 A large collection of congruent right triangles is given, each with side length 3,4,5. Find the maximal number of such triangles you can place inside a 20x20 square, with no two triangles intersecting (in their interiors).

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4 Three rational number x, p, q satisfy $p^2 - xq^2$=1. Prove that there are integers a, b such that $p = \frac{a^2+xb^2}{a^2-xb^2}$ and $q = \frac{2ab}{a^2-xb^2}$.

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5 A regular pentagon $ABCDE$ is given. The point X is on his circumcircle, on the arc $\overset{\frown}{AE}$. Prove that $|AX| + |CX| + |EX| = |BX| + |DX|$.

Remark: Here's a more general version of the problem: Prove that for any point X in the plane, $|AX| + |CX| + |EX| \ge |BX| + |DX|$, with equality only on the arc $\overset{\frown}{AE}$.

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6 Let $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ be a function satisfying:

- For any $x_1, x_2, y_1, y_2 \in \mathbb{Q}$,

$$f\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \le \frac{f(x_1, y_1) + f(x_2, y_2)}{2}.$$

- $f(0, 0) \le 0$.
- For any $x, y \in \mathbb{Q}$ satisfying $x^2 + y^2 > 100$, the inequality $f(x, y) > 1$ holds.

Prove that there is some positive rational number b such that for all rationals x, y,

$$f(x,y) \geq b\sqrt{x^2+y^2} - \frac{1}{b}.$$

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7 A table with m rows and n columns is given. In each cell of the table an integer is written. Heisuke and Oscar play the following game: at the beginning of each turn, Heisuke may choose to swap any two columns. Then he chooses some rows and writes down a new row at the bottom of the table, with each cell consisting the sum of the corresponding cells in the chosen rows. Oscar then deletes one row chosen by Heisuke (so that at the end of each turn there are exactly m rows). Then the next turn begins and so on. Prove that Heisuke can assure that, after some finite amount of turns, no number in the table is smaller than the number to the number on his right.
Example: If we begin with $(1,1,1),(6,5,4),(9,8,7)$, Heisuke may choose to swap the first and third column to get $(1,1,1),(4,5,6),(7,8,9)$. Then he chooses the first and second rows to obtain $(1,1,1),(4,5,6),(7,8,9),(5,6,7)$. Then Oscar has to delete either the first or the second row, let's say the second. We get $(1,1,1),(7,8,9),(5,6,7)$ and Heisuke wins.
