NIMO Summer Contest 2012

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by AIME15

1 Let $f(x) = (x^4 + 2x^3 + 4x^2 + 2x + 1)^5$. Compute the prime p satisfying $f(p) = 418{,}195{,}493$.

Proposed by Eugene Chen

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2 Compute the number of positive integers n satisfying the inequalities

$$2^{n-1} < 5^{n-3} < 3^n.$$

Proposed by Isabella Grabski

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3 Let

$$S = \sum_{i=1}^{2012} i!.$$

The tens and units digits of S (in decimal notation) are a and b, respectively. Compute $10a + b$.

Proposed by Lewis Chen

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4 The degree measures of the angles of nondegenerate hexagon $ABCDEF$ are integers that form a non-constant arithmetic sequence in some order, and $\angle A$ is the smallest angle of the (not necessarily convex) hexagon. Compute the sum of all possible degree measures of $\angle A$.

Proposed by Lewis Chen

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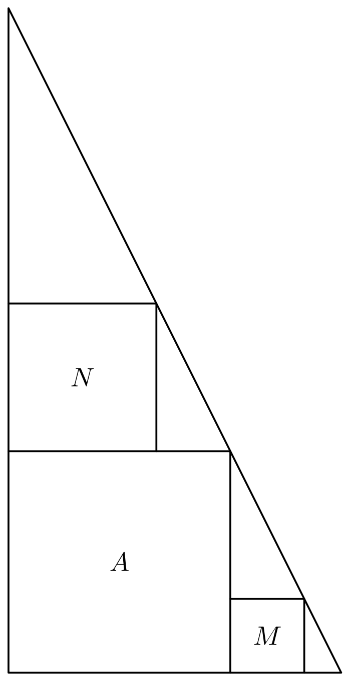
5 In the diagram below, three squares are inscribed in right triangles. Their areas are A, M, and N, as indicated in the diagram. If $M = 5$ and $N = 12$, then A can be expressed as $a + b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Compute $a + b + c$.



Proposed by Aaron Lin

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6 When Eva counts, she skips all numbers containing a digit divisible by 3. For example, the first ten numbers she counts are 1, 2, 4, 5, 7, 8, 11, 12, 14, 15. What is the 100^{th} number she counts?

Proposed by Eugene Chen

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7 A permutation $(a_1, a_2, a_3, \dots, a_{2012})$ of $(1, 2, 3, \dots, 2012)$ is selected at random. If S is the expected value of

$$\sum_{i=1}^{2012} |a_i - i|,$$

then compute the sum of the prime factors of S.

Proposed by Aaron Lin

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8 Points A, B, and O lie in the plane such that $\measuredangle AOB = 120^\circ$. Circle ω_0 with radius 6 is constructed tangent to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$. For all $i \ge 1$, circle ω_i with radius r_i is constructed such that $r_i < r_{i-1}$ and ω_i is tangent to $\overrightarrow{OA}$, $\overrightarrow{OB}$, and ω_{i-1}. If

$$S = \sum_{i=1}^{\infty} r_i,$$

then S can be expressed as $a\sqrt{b}+c$, where a, b, c are integers and b is not divisible by the square of any prime. Compute $100a + 10b + c$.

Proposed by Aaron Lin

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9 A quadratic polynomial $p(x)$ with integer coefficients satisfies $p(41) = 42$. For some integers $a, b > 41$, $p(a) = 13$ and $p(b) = 73$. Compute the value of $p(1)$.

Proposed by Aaron Lin

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10 A *triangulation* of a polygon is a subdivision of the polygon into triangles meeting edge to edge, with the property that the set of triangle vertices coincides with the set of vertices of the polygon. Adam randomly selects a triangulation of a regular 180-gon. Then, Bob selects one of the 178 triangles in this triangulation. The expected number of 1° angles in this triangle can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

Proposed by Lewis Chen

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11 Let a and b be two positive integers satisfying the equation

$$20\sqrt{12} = a\sqrt{b}.$$

Compute the sum of all possible distinct products ab.

Proposed by Lewis Chen

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12 The NEMO (National Electronic Math Olympiad) is similar to the NIMO Summer Contest, in that there are fifteen problems, each worth a set number of points. However, the NEMO is weighted using Fibonacci numbers; that is, the n^{th} problem is worth F_n points, where $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. The two problem writers are fair people, so they make sure that each of them is responsible for problems worth an equal number of total points. Compute the number of ways problem writing assignments can be distributed between the two writers.

Proposed by Lewis Chen

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13 For the NEMO, Kevin needs to compute the product

$$9 \times 99 \times 999 \times \cdots \times 999999999.$$

Kevin takes exactly ab seconds to multiply an a-digit integer by a b-digit integer. Compute the minimum number of seconds necessary for Kevin to evaluate the expression together by performing eight such multiplications.

Proposed by Evan Chen

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14 A set of lattice points is called *good* if it does not contain two points that form a line with slope -1 or slope 1. Let $S = \{(x, y) \mid x, y \in \mathbb{Z}, 1 \leq x, y \leq 4\}$. Compute the number of non-empty good subsets of S.

Proposed by Lewis Chen

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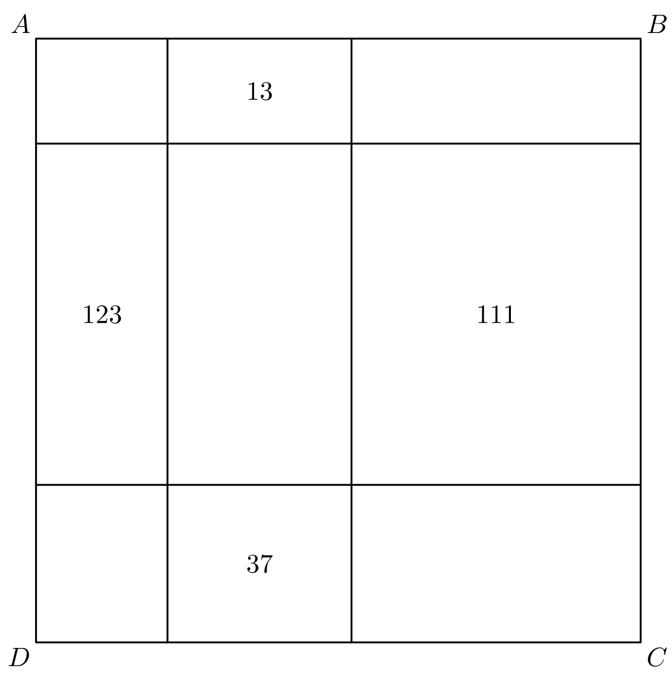
15 In the diagram below, square $ABCD$ with side length 23 is cut into nine rectangles by two lines parallel to $\overline{AB}$ and two lines parallel to $\overline{BC}$. The areas of four of these rectangles are indicated in the diagram. Compute the largest possible value for the area of the central rectangle.



Proposed by Lewis Chen
