NIMO Summer Contest 2013

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1 What is the maximum possible score on this contest? Recall that on the NIMO 2013 Summer Contest, problems $1, 2, \dots, 15$ are worth $1, 2, \dots, 15$ points, respectively.

Proposed by Evan Chen

2 If $\frac{2+4+6}{1+3+5} - \frac{1+3+5}{2+4+6} = \frac{m}{n}$ for relatively prime integers m and n, compute $100m + n$.

Proposed by Evan Chen

3 Jacob and Aaron are playing a game in which Aaron is trying to guess the outcome of an unfair coin which shows heads $\frac{2}{3}$ of the time. Aaron randomly guesses "heads" $\frac{2}{3}$ of the time, and guesses "tails" the other $\frac{1}{3}$ of the time. If the probability that Aaron guesses correctly is p, compute $9000p$.

Proposed by Aaron Lin

4 Find the sum of the real roots of the polynomial

$$\prod_{k=1}^{100} \left(x^2 - 11x + k\right) = \left(x^2 - 11x + 1\right)\left(x^2 - 11x + 2\right) \dots \left(x^2 - 11x + 100\right).$$

Proposed by Evan Chen

5 A point (a, b) in the plane is called *sparkling* if it also lies on the line $ax + by = 1$. Find the maximum possible distance between two sparkling points.

Proposed by Evan Chen

6 Let ABC and DEF be two triangles, such that $AB = DE = 20$, $BC = EF = 13$, and $\angle A = \angle D$. If $AC - DF = 10$, determine the area of $\triangle ABC$.

Proposed by Lewis Chen

7 Circle ω_1 and ω_2 have centers $(0, 6)$ and $(20, 0)$, respectively. Both circles have radius 30, and intersect at two points X and Y. The line through X and Y can be written in the form $y = mx + b$. Compute $100m + b$.

Proposed by Evan Chen

8 A pair of positive integers (m, n) is called *compatible* if $m \geq \frac{1}{2}n + 7$ and $n \geq \frac{1}{2}m + 7$. A positive integer $k \geq 1$ is called *lonely* if (k, ℓ) is not compatible for any integer $\ell \geq 1$. Find the sum of all lonely integers.

Proposed by Evan Chen

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9 Compute $99(99^2 + 3) + 3 \cdot 99^2$.

Proposed by Evan Chen

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10 Let $P(x)$ be the unique polynomial of degree four for which $P(165) = 20$, and

$$P(42) = P(69) = P(96) = P(123) = 13.$$

Compute $P(1) - P(2) + P(3) - P(4) + \cdots + P(165)$.

Proposed by Evan Chen

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11 Find $100m + n$ if m and n are relatively prime positive integers such that

$$\sum_{\substack{i,j \geq 0 \\ i+j \text{ odd}}} \frac{1}{2^i 3^j} = \frac{m}{n}.$$

Proposed by Aaron Lin

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12 In $\triangle ABC$, $AB = 40$, $BC = 60$, and $CA = 50$. The angle bisector of $\angle A$ intersects the circumcircle of $\triangle ABC$ at A and P. Find BP.

Proposed by Eugene Chen

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13 In trapezoid $ABCD$, $AD \parallel BC$ and $\angle ABC + \angle CDA = 270^\circ$. Compute AB^2 given that $AB \cdot \tan(\angle BCD) = 20$ and $CD = 13$.

Proposed by Lewis Chen

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14 Let p, q, and r be primes satisfying

$$pqr = 18999999999999999999999999999999999999999999999999999962.$$

Compute $S(p) + S(q) + S(r) - S(pqr)$, where $S(n)$ denote the sum of the decimals digits of n.

Proposed by Evan Chen

15

Ted quite likes haikus,

poems with five-seven-five,

but Ted knows few words.

He knows $2n$ words

that contain n syllables

for every int n.

Ted can only write
N distinct haikus. Find N.

Take mod one hundred.

Ted loves creating haikus (Japanese three-line poems with 5, 7, 5 syllables each), but his vocabulary is rather limited. In particular, for integers $1 \leq n \leq 7$, he knows $2n$ words with n syllables. Furthermore, words cannot cross between lines, but may be repeated. If Ted can make N distinct haikus, compute the remainder when N is divided by 100.

Proposed by Lewis Chen