NIMO Summer Contest 2014

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by v_Enhance

– August 23rd

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1 Compute $1 + 2 \cdot 3^4$.

Proposed by Evan Chen

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2 How many $2 \times 2 \times 2$ cubes must be added to a $8 \times 8 \times 8$ cube to form a $12 \times 12 \times 12$ cube?

Proposed by Evan Chen

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3 A square and equilateral triangle have the same perimeter. If the triangle has area $16\sqrt{3}$, what is the area of the square?

Proposed by Evan Chen

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4 Let n be a positive integer. Determine the smallest possible value of $1 - n + n^2 - n^3 + \dots + n^{1000}$.

Proposed by Evan Chen

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5 We have a five-digit positive integer N. We select every pair of digits of N (and keep them in order) to obtain the $\binom{5}{2} = 10$ numbers 33, 37, 37, 37, 38, 73, 77, 78, 83, 87. Find N.

Proposed by Lewis Chen

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6 Suppose x is a random real number between 1 and 4, and y is a random real number between 1 and 9. If the expected value of

$$\lceil \log_2 x \rceil - \lfloor \log_3 y \rfloor$$

can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers, compute $100m + n$.

Proposed by Lewis Chen

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7 Evaluate

$$\frac{1}{729} \sum_{a=1}^{9} \sum_{b=1}^{9} \sum_{c=1}^{9} \left(abc + ab + bc + ca + a + b + c \right).$$

Proposed by Evan Chen

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8 Aaron takes a square sheet of paper, with one corner labeled A. Point P is chosen at random inside of the square and Aaron folds the paper so that points A and P coincide. He cuts the sheet along the crease and discards the piece containing A. Let p be the probability that the remaining piece is a pentagon. Find the integer nearest to $100p$.

Proposed by Aaron Lin

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9 Two players play a game involving an $n \times n$ grid of chocolate. Each turn, a player may either eat a piece of chocolate (of any size), or split an existing piece of chocolate into two rectangles along a grid-line. The player who moves last loses. For how many positive integers n less than 1000 does the second player win?

(Splitting a piece of chocolate refers to taking an $a \times b$ piece, and breaking it into an $(a-c) \times b$ and a $c \times b$ piece, or an $a \times (b-d)$ and an $a \times d$ piece.)

Proposed by Lewis Chen

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10 Among 100 points in the plane, no three collinear, exactly 4026 pairs are connected by line segments. Each point is then randomly assigned an integer from 1 to 100 inclusive, each equally likely, such that no integer appears more than once. Find the expected value of the number of segments which join two points whose labels differ by at least 50.

Proposed by Evan Chen

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11 Consider real numbers A, B, $\dots$, Z such that

$$EVIL = \frac{5}{31}, \; LOVE = \frac{6}{29}, \text{ and } IMO = \frac{7}{3}.$$

If $OMO = \frac{m}{n}$ for relatively prime positive integers m and n, find the value of $m+n$.

Proposed by Evan Chen

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12 Find the sum of all positive integers n such that

$$\frac{2n+1}{n(n-1)}$$

has a terminating decimal representation.

Proposed by Evan Chen

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13 Let α and β be nonnegative integers. Suppose the number of strictly increasing sequences of integers $a_0, a_1, \dots, a_{2014}$ satisfying $0 \le a_m \le 3m$ is $2^\alpha(2\beta+1)$. Find α.

Proposed by Lewis Chen

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14 Let ABC be a triangle with circumcenter O and let X, Y, Z be the midpoints of arcs BAC, ABC, ACB on its circumcircle. Let G and I denote the centroid of $\triangle XYZ$ and the incenter of $\triangle ABC$.

Given that $AB = 13$, $BC = 14$, $CA = 15$, and $\frac{GO}{GI} = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Evan Chen

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15 Let $A = (0, 0)$, $B = (-1, -1)$, $C = (x, y)$, and $D = (x + 1, y)$, where $x > y$ are positive integers. Suppose points A, B, C, D lie on a circle with radius r. Denote by r_1 and r_2 the smallest and second smallest possible values of r. Compute $r_1^2 + r_2^2$.

Proposed by Lewis Chen
