Princeton University Math Competition 2006

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by hshiems

1 Given that $x^2+5x+6=20$, find the value of $3x^2+15x+17$.

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2 Express $\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}$ in the simplest possible form.

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3 Let $r_1,\dots,r_5$ be the roots of the polynomial $x^5+5x^4-79x^3+64x^2+60x+144$. What is $r_1^2+\dots+r_5^2$?

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4 Find all pairs of real numbers (a,b) so that there exists a polynomial $P(x)$ with real coefficients and $P(P(x))=x^4-8x^3+ax^2+bx+40$.

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5 Find the greatest integer less than the number

$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{1000000}}$

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6 Suppose that $P(x)$ is a polynomial with the property that there exists another polynomial $Q(x)$ to satisfy $P(x)Q(x)=P(x^2)$. $P(x)$ and $Q(x)$ may have complex coefficients. If $P(x)$ is a quintic with distinct complex roots $r_1,\dots,r_5$, find all possible values of $|r_1|+\dots+|r_5|$.

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7 Find one complex value of x that satisfies the equation $\sqrt{3}x^7+x^4+2=0$.

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8 The Lucas numbers L_n are defined recursively as follows: $L_0=2, L_1=1, L_n=L_{n-1}+L_{n-2}$ for $n\geq 2$. Let $r=0.21347\dots$, whose digits form the pattern of the Lucas numbers. When the numbers have multiple digits, they will "overlap," so $r=0.2134830\dots$, **not** $0.213471118\dots$. Express r as a rational number $\frac{p}{q}$, where p and q are relatively prime.

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9 The curve $y=x^4+2x^3-11x^2-13x+35$ has a bitangent (a line tangent to the curve at two points). What is the equation of the bitangent?

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10 If x,y,z are real numbers and

$$\begin{aligned} 2x + y + z &\leq 66 \\ x + 2y + z &\leq 60 \\ x + y + 2z &\leq 70 \\ x + 2y + 3z &\leq 110 \\ 3x + y + 2z &\leq 98 \\ 2x + 3y + z &\leq 89 \end{aligned}$$

What is the maximum possible value of $x + y + z$?