Princeton University Math Competition 2013

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– Algebra A

1 Suppose $a, b, c > 0$ are integers such that

$$abc - bc - ac - ab + a + b + c = 2013.$$

Find the number of possibilities for the ordered triple (a, b, c).

2 Find the number of pairs (n, C) of positive integers such that $C \leq 100$ and $n^2 + n + C$ is a perfect square.

3 Let $x_1 = \sqrt{10}$ and $y_1 = \sqrt{3}$. For all $n \geq 2$, let

$$x_n = x_{n-1}\sqrt{77} + 15y_{n-1}$$
$$y_n = 5x_{n-1} + y_{n-1}\sqrt{77}$$

Find $x_5^6 + 2x_5^4 - 9x_5^4y_5^2 - 12x_5^2y_5^2 + 27x_5^2y_5^4 + 18y_5^4 - 27y_5^6$.

4 Suppose a, b are nonzero integers such that two roots of $x^3 + ax^2 + bx + 9a$ coincide, and all three roots are integers. Find $|ab|$.

5 Suppose w, x, y, z satisfy

$$w + x + y + z = 25,$$
$$wx + wy + wz + xy + xz + yz = 2y + 2z + 193$$

The largest possible value of w can be expressed in lowest terms as w_1/w_2 for some integers $w_1, w_2 > 0$. Find $w_1 + w_2$.

6 Suppose the function ψ satisfies $\psi(1) = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$ and $\psi(3x) + 3\psi(x) = \psi(x)^3$ for all real x. Determine the greatest integer less than $\prod_{n=1}^{100} \psi(3^n)$.

7 Evaluate

$$\sqrt{2013 + 276\sqrt{2027 + 278\sqrt{2041 + 280\sqrt{2055 + \ldots}}}}$$

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8 Let $\mathcal{S}$ be the set of permutations of $\{1, 2, \ldots, 6\}$, and let $\mathcal{T}$ be the set of permutations of $\mathcal{S}$ that preserve compositions: i.e., if $F \in \mathcal{T}$ then

$$F(f_2 \circ f_1) = F(f_2) \circ F(f_1)$$

for all $f_1, f_2 \in \mathcal{S}$. Find the number of elements $F \in \mathcal{T}$ such that if $f \in \mathcal{S}$ satisfies $f(1) = 2$ and $f(2) = 1$, then $(F(f))(1) = 2$ and $(F(f))(2) = 1$.

– Algebra B

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1 Suppose $a, b, c > 0$ are integers such that

$$abc - bc - ac - ab + a + b + c = 2013.$$

Find the number of possibilities for the ordered triple (a, b, c).

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2 Betty Lou and Peggy Sue take turns flipping switches on a 100×100 grid. Initially, all switches are "off". Betty Lou always flips a horizontal row of switches on her turn; Peggy Sue always flips a vertical column of switches. When they finish, there is an odd number of switches turned "on" in each row and column. Find the maximum number of switches that can be on, in total, when they finish.

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3 Let $x_1 = 1/20$, $x_2 = 1/13$, and

$$x_{n+2} = \frac{2x_n x_{n+1}(x_n + x_{n+1})}{x_n^2 + x_{n+1}^2}$$

for all integers $n \geq 1$. Evaluate $\sum_{n=1}^{\infty}(1/(x_n + x_{n+1}))$.

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4 Let $f(x) = 1 - |x|$. Let

$$f_n(x) = (\overbrace{f \circ \cdots \circ f}^{n \text{ copies}})(x)$$
$$g_n(x) = |n - |x||$$

Determine the area of the region bounded by the x-axis and the graph of the function $\sum_{n=1}^{10} f(x) + \sum_{n=1}^{10} g(x)$.

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5 Find the number of pairs (n, C) of positive integers such that $C \leq 100$ and $n^2 + n + C$ is a perfect square.

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6 Suppose a, b are nonzero integers such that two roots of $x^3 + ax^2 + bx + 9a$ coincide, and all three roots are integers. Find $|ab|$.

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7 Evaluate

$$\sqrt{2013+276\sqrt{2027+278\sqrt{2041+280\sqrt{2055+\dots}}}}$$

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8 If x, y are real, then the *absolute value* of the complex number $z = x + yi$ is

$$|z| = \sqrt{x^2 + y^2}.$$

Find the number of polynomials $f(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + t^4$ such that $A_0, \dots, A_3$ are integers and all roots of f in the complex plane have absolute value ≤ 1.

– Combinatorics A

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1 A regular pentagon can have the line segments forming its boundary extended to lines, giving an arrangement of lines that intersect at ten points. How many ways are there to choose five points of these ten so that no three of the points are collinear?

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2 How many ways are there to color the edges of a hexagon orange and black if we assume that two hexagons are indistinguishable if one can be rotated into the other? Note that we are saying the colorings OOBBOB and BOBBOO are distinct; we ignore flips.

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3 How many tuples of integers $(a_0, a_1, a_2, a_3, a_4)$ are there, with $1 \leq a_i \leq 5$ for each i, so that $a_0 < a_1 > a_2 < a_3 > a_4$?

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4 You roll three fair six-sided dice. Given that the highest number you rolled is a 5, the expected value of the sum of the three dice can be written as $\frac{a}{b}$ in simplest form. Find $a + b$.

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5 Mereduth has many red boxes and many blue boxes. Coloon has placed five green boxes in a row on the ground, and Mereduth wants to arrange some number of her boxes on top of his row. Assume that each box must be placed so that it straddles two lower boxes. Including the one with no boxes, how many arrangements can Mereduth make?

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6 A sequence of vertices $v_1, v_2, \dots, v_k$ in a graph, where $v_i = v_j$ only if $i = j$ and k can be any positive integer, is called a *cycle* if v_1 is attached by an edge to v_2, v_2 to v_3, and so on to v_k connected to v_1. Rotations and reflections are distinct: A, B, C is distinct from A, C, B and B, C, A. Supposed a simple graph G has 2013 vertices and 3013 edges. What is the minimal number of cycles possible in G?

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7 The Miami Heat and the San Antonio Spurs are playing a best-of-five series basketball championship, in which the team that first wins three games wins the whole series. Assume that the probability that the Heat wins a given game is x (there are no ties). The expected value for the total number of games played can be written as $f(x)$, with f a polynomial. Find $f(-1)$.

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8 Eight all different sushis are placed evenly on the edge of a round table, whose surface can rotate around the center. Eight people also evenly sit around the table, each with one sushi in front. Each person has one favorite sushi among these eight, and they are all distinct. They find that no matter how they rotate the table, there are never more than three people who have their favorite sushis in front of them simultaneously. By this requirement, how many different possible arrangements of the eight sushis are there? Two arrangements that differ by a rotation are considered the same.

– Combinatorics B

1 Including the original, how many ways are there to rearrange the letters in PRINCETON so that no two vowels (I, E, O) are consecutive and no three consonants (P, R, N, C, T, N) are consecutive?

2 The number of positive integer pairs (a, b) that have a dividing b and b dividing 2013^{2014} can be written as $2013n + k$, where n and k are integers and $0 \le k < 2013$. What is k? Recall $2013 = 3 \cdot 11 \cdot 61$.

3 Chris's pet tiger travels by jumping north and east. Chris wants to ride his tiger from Fine Hall to McCosh, which is 3 jumps east and 10 jumps north. However, Chris wants to avoid the horde of PUMaC competitors eating lunch at Frist, located 2 jumps east and 4 jumps north of Fine Hall. How many ways can he get to McCosh without going through Frist?

4 Mereduth has many red boxes and many blue boxes. Coloon has placed five green boxes in a row on the ground, and Mereduth wants to arrange some number of her boxes on top of his row. Assume that each box must be placed so that it straddles two lower boxes. Including the one with no boxes, how many arrangements can Mereduth make?

5 A sequence of vertices $v_1, v_2, \ldots, v_k$ in a graph, where $v_i = v_j$ only if $i = j$ and k can be any positive integer, is called a *cycle* if v_1 is attached by an edge to v_2, v_2 to v_3, and so on to v_k connected to v_1. Rotations and reflections are distinct: A, B, C is distinct from A, C, B and B, C, A. Supposed a simple graph G has 2013 vertices and 3013 edges. What is the minimal number of cycles possible in G?

6 An integer sequence $a_1, a_2, \ldots, a_n$ has $a_1 = 0$, $a_n \le 10$ and $a_{i+1} - a_i \ge 2$ for $1 \le i < n$. How many possibilities are there for this sequence? The sequence may be of any length.

7 You are eating at a fancy restaurant with a person you wish to impress. For some reason, you think that eating at least one spicy course and one meat-filled course will impress the person. The meal is five courses, with four options for each course. Each course has one option that is spicy and meat-filled, one option that is just spicy, one that is just meat-filled, and one that is neither spicy nor meat-filled. How many possible meals can you have?

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8	You roll three fair six-sided dice. Given that the highest number you rolled is a 5, the expected value of the sum of the three dice can be written as $\frac{a}{b}$ in simplest form. Find $a+b$.	
–	Geometry A	
1	Let O be a point with three other points A, B, C and $\angle AOB = \angle BOC = \angle AOC = 2\pi/3$. Consider the average area of the set of triangles ABC where $OA, OB, OC \in \{3, 4, 5\}$. The average area can be written in the form $m\sqrt{n}$ where m, n are integers and n is not divisible by a perfect square greater than 1. Find $m+n$.	
2	An equilateral triangle is given. A point lies on the incircle of this triangle. If the smallest two distances from the point to the sides of the triangle is 1 and 4, the sidelength of this equilateral triangle can be expressed as $\frac{a\sqrt{b}}{c}$ where $(a, c) = 1$ and b is not divisible by the square of an integer greater than 1. Find $a+b+c$.	
3	Consider the shape formed from taking equilateral triangle ABC with side length 6 and tracing out the arc BC with center A. Set the shape down on line l so that segment AB is perpendicular to l, and B touches l. Beginning from arc BC touching l, we roll ABC along l until both points A and C are on the line. The area traced out by the roll can be written in the form $n\pi$, where n is an integer. Find n.	
4	Draw an equilateral triangle with center O. Rotate the equilateral triangle $30^\circ, 60^\circ, 90^\circ$ with respect to O so there would be four congruent equilateral triangles on each other. Look at the diagram. If the smallest triangle has area 1, the area of the original equilateral triangle could be expressed as $p+q\sqrt{r}$ where p, q, r are positive integers and r is not divisible by a square greater than 1. Find $p+q+r$.	
5	Suppose you have a sphere tangent to the xy-plane with its center having positive z-coordinate. If it is projected from a point $P = (0, b, a)$ to the xy-plane, it gives the conic section $y = x^2$. If we write $a = \frac{p}{q}$ where p, q are integers, find $p+q$.	
6	On a circle, points A, B, C, D lie counterclockwise in this order. Let the orthocenters of ABC, BCD, CDA, I be H, I, J, K respectively. Let $HI = 2$, $IJ = 3$, $JK = 4$, $KH = 5$. Find the value of $13(BD)^2$.	
7	Given triangle ABC and a point P inside it, $\angle BAP = 18^\circ$, $\angle CAP = 30^\circ$, $\angle ACP = 48^\circ$, and $AP = BC$. If $\angle BCP = x^\circ$, find x.	
8	Three chords of a sphere, each having length $5, 6, 7$, intersect at a single point inside the sphere and are pairwise perpendicular. For R the maximum possible radius of this sphere, find R^2.	
–	Geometry B	

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1 We construct three circles: O with diameter AB and area $12+2x$, P with diameter AC and area $24+x$, and Q with diameter BC and area $108-x$. Given that C is on circle O, compute x.

2 Triangle ABC satisfies $\angle ABC = \angle ACB = 78^\circ$. Points D and E lie on AB, AC and satisfy $\angle BCD = 24^\circ$ and $\angle CBE = 51^\circ$. If $\angle BED = x^\circ$, find x.

3 Consider all planes through the center of a $2\times 2\times 2$ cube that create cross sections that are regular polygons. The sum of the cross sections for each of these planes can be written in the form $a\sqrt{b}+c$, where b is a square-free positive integer. Find $a+b+c$.

4 An equilateral triangle is given. A point lies on the incircle of this triangle. If the smallest two distances from the point to the sides of the triangle is 1 and 4, the sidelength of this equilateral triangle can be expressed as $\frac{a\sqrt{b}}{c}$ where $(a,c)=1$ and b is not divisible by the square of an integer greater than 1. Find $a+b+c$.

5 Circle w with center O meets circle Γ at X, Y, and O is on Γ. Point $Z\in\Gamma$ lies outside w such that $XZ = 11$, $OZ = 15$, and $YZ = 13$. If the radius of circle w is r, find r^2.

6 Draw an equilateral triangle with center O. Rotate the equilateral triangle $30^\circ, 60^\circ, 90^\circ$ with respect to O so there would be four congruent equilateral triangles on each other. Look at the diagram. If the smallest triangle has area 1, the area of the original equilateral triangle could be expressed as $p+q\sqrt{r}$ where p, q, r are positive integers and r is not divisible by a square greater than 1. Find $p+q+r$.

7 A tetrahedron $ABCD$ satisfies $AB = 6$, $CD = 8$, and $BC = DA = 5$. Let V be the maximum value of $ABCD$ possible. If we can write $V^4 = 2^n 3^m$ for some integers m and n, find mn.

8 Triangle $A_1B_1C_1$ is an equilateral triangle with sidelength 1. For each $n > 1$, we construct triangle $A_nB_nC_n$ from $A_{n-1}B_{n-1}C_{n-1}$ according to the following rule: A_n, B_n, C_n are points on segments $A_{n-1}B_{n-1}, B_{n-1}C_{n-1}, C_{n-1}A_{n-1}$ respectively, and satisfy the following:

$$\frac{A_{n-1}A_n}{A_nB_{n-1}} = \frac{B_{n-1}B_n}{B_nC_{n-1}} = \frac{C_{n-1}C_n}{C_nA_{n-1}} = \frac{1}{n-1}$$

So for example, $A_2B_2C_2$ is formed by taking the midpoints of the sides of $A_1B_1C_1$. Now, we can write $\frac{|A_5B_5C_5|}{|A_1B_1C_1|} = \frac{m}{n}$ where m and n are relatively prime integers. Find $m+n$. (For a triangle $\triangle ABC$, $|ABC|$ denotes its area.)

– Number Theory A

1 If p, q, and r are primes with $pqr = 7(p+q+r)$, find $p+q+r$.

2 What is the smallest positive integer n such that 2013^n ends in 001 (i.e. the rightmost three digits of 2013^n are 001?

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3 Let A be the greatest possible value of a product of positive integers that sums to 2014. Compute the sum of all bases and exponents in the prime factorization of A. For example, if $A = 7 \cdot 11^5$, the answer would be $7 + 11 + 5 = 23$.

4 Let d be the greatest common divisor of $2^{30^{10}} - 2$ and $2^{30^{45}} - 2$. Find the remainder when d is divided by 2013.

5 Define a "digitized number" as a ten-digit number $a_0a_1 \ldots a_9$ such that for $k = 0, 1, \ldots, 9$, a_k is equal to the number of times the digit k occurs in the number. Find the sum of all digitized numbers.

6 What is the largest positive integer that cannot be expressed as a sum of non-negative integer multiple of 13, 17, and 23?

7 Suppose $P(x)$ is a degree n monic polynomial with integer coefficients such that 2013 divides $P(r)$ for exactly 1000 values of r between 1 and 2013 inclusive. Find the minimum value of n.

8 Find the number of primes p between 100 and 200 for which $x^{11} + y^{16} \equiv 2013 \pmod p$ has a solution in integers x and y.

– Number Theory B

1 If p, q, and r are primes with $pqr = 7(p + q + r)$, find $p + q + r$.

2 What is the smallest positive integer n such that 2013^n ends in 001 (i.e. the rightmost three digits of 2013^n are 001?

3 Find the smallest positive integer x such that

- x is 1 more than a multiple of 3,
- x is 3 more than a multiple of 5,
- x is 5 more than a multiple of 7,
- x is 9 more than a multiple of 11, and
- x is 2 more than a multiple of 13.

4 Compute the smallest integer $n \geq 4$ such that $\binom{n}{4}$ ends in 4 or more zeroes (i.e. the rightmost four digits of $\binom{n}{4}$ are 0000).

5 Let A be the greatest possible value of a product of positive integers that sums to 2014. Compute the sum of all bases and exponents in the prime factorization of A. For example, if $A = 7 \cdot 11^5$, the answer would be $7 + 11 + 5 = 23$.

6 Let d be the greatest common divisor of $2^{30^{10}} - 2$ and $2^{30^{45}} - 2$. Find the remainder when d is divided by 2013.

7 Define a "digitized number" as a ten-digit number $a_0a_1\dots a_9$ such that for $k = 0, 1, \dots, 9$, a_k is equal to the number of times the digit k occurs in the number. Find the sum of all digitized numbers.

8 What is the largest positive integer that cannot be expressed as a sum of non-negative integer multiple of 13, 17, and 23?

– Individual Finals A

1 Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2} \le \frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$

for all positive real numbers a, b and c satisfying $a^2 + b^2 + c^2 = 1$.

2 Let γ be the incircle of $\triangle ABC$ (i.e. the circle inscribed in $\triangle ABC$) and I be the center of γ. Let D, E and F be the feet of the perpendiculars from I to BC, CA, and AB respectively. Let D' be the point on γ such that DD' is a diameter of γ. Suppose the tangent to γ through D intersects the line EF at P. Suppose the tangent to γ through D' intersects the line EF at Q. Prove that $\angle PIQ + \angle DAD' = 180°$.

3 A graph consists of a set of vertices, some of which are connected by (undirected) edges. A *star* of a graph is a set of edges with a common endpoint. A *matching* of a graph is a set of edges such that no two have a common endpoint. Show that if the number of edges of a graph G is larger than $2(k-1)^2$, then G contains a matching of size k or a star of size k.

– Individual Finals B

1 Let $a_1 = 2013$ and $a_{n+1} = 2013^{a_n}$ for all positive integers n. Let $b_1 = 1$ and $b_{n+1} = 2013^{2012b_n}$ for all positive integers n. Prove that $a_n > b_n$ for all positive integers n.

2 Find all pairs of positive integers (a, b) such that:

$$\frac{a^3+4b}{a+2b^2+2a^2b}$$

is a positive integer.

3 Find the smallest positive integer n with the following property: for every sequence of positive integers $a_1, a_2, \dots, a_n$ with $a_1+a_2+\dots+a_n = 2013$, there exist some (possibly one) consecutive term(s) in the sequence that add up to 70.

–	Team
1	A token is placed in the leftmost square in a strip of four squares. In each move, you are allowed to move the token left or right along the strip by sliding it a single square, provided that the token stays on the strip. In how many ways can the token be moved so that after exactly 15 moves, it is in the rightmost square of the strip?
2	(Following question 1) Now instead consider an infinite strip of squares, labeled with the integers $0, 1, 2, \ldots$ in that order. You start at the square labeled 0. You want to end up at the square labeled 3. In how many ways can this be done in exactly 15 moves?
3	The area of a circle centered at the origin, which is inscribed in the parabola $y = x^2 - 25$, can be expressed as $\frac{a}{b}\pi$, where a and b are coprime positive integers. What is the value of $a + b$?
4	Find the sum of all positive integers m such that 2^m can be expressed as a sum of four factorials (of positive integers). Note: The factorials do not have to be distinct. For example, $2^4 = 16$ counts, because it equals $3! + 3! + 2! + 2!$.
5	A palindrome number is a positive integer that reads the same forward and backward. For example, 1221 and 8 are palindrome numbers whereas 69 and 157 are not. A and B are 4-digit palindrome numbers. C is a 3-digit palindrome number. Given that $A - B = C$, what is the value of C?
6	How many positive integers n less than 1000 have the property that the number of positive integers less than n which are coprime to n is exactly $\frac{n}{3}$?
7	Find the total number of triples of integers (x, y, n) satisfying the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{n^2}$, where n is either 2012 or 2013.
8	Let k be a positive integer with the following property: For every subset A of $\{1, 2, \ldots, 25\}$ with $\|A\| = k$, we can find distinct elements x and y of A such that $\frac{2}{3} \leq \frac{x}{y} \leq \frac{3}{2}$. Find the smallest possible value of k.
9	If two distinct integers from 1 to 50 inclusive are chosen at random, what is the expected value of their product? Note: The expectation is defined as the sum of the products of probability and value, i.e., the expected value of a coin flip that gives you $\$10$ if head and $\$5$ if tail is $\frac{1}{2} \times \$10 + \frac{1}{2} \times \$5 = \$7.5$.
10	On a plane, there are 7 seats. Each is assigned to a passenger. The passengers walk on the plane one at a time. The first passenger sits in the wrong seat (someone else's). For all the

following people, they either sit in their assigned seat, or if it is full, randomly pick another. You are the last person to board the plane. What is the probability that you sit in your own seat?

11 If two points are selected at random on a fixed circle and the chord between the two points is drawn, what is the probability that its length exceeds the radius of the circle?

12 Let D be a point on the side BC of $\triangle ABC$. If $AB = 8$, $AC = 7$, $BD = 2$, and $CD = 1$, find AD.

13 The equation $x^5 - 2x^4 - 1 = 0$ has five complex roots r_1, r_2, r_3, r_4, r_5. Find the value of

$$\frac{1}{r_1^8} + \frac{1}{r_2^8} + \frac{1}{r_3^8} + \frac{1}{r_4^8} + \frac{1}{r_5^8}.$$

14 Shuffle a deck of 71 playing cards which contains 6 aces. Then turn up cards from the top until you see an ace. What is the average number of cards required to be turned up to find the first ace?

15 Prove:

$$|\sin a_1| + |\sin a_2| + |\sin a_3| + \ldots + |\sin a_n| + |\cos(a_1 + a_2 + a_3 + \ldots + a_n)| \geq 1.$$

16 Is $\cos 1^\circ$ rational? Prove.

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