Math Prize For Girls Problems 2011

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by Ravi B

1 If m and n are integers such that $3m+4n=100$, what is the smallest possible value of $|m-n|$?

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2 Express $\sqrt{2+\sqrt{3}}$ in the form $\frac{a+\sqrt{b}}{\sqrt{c}}$, where a is a positive integer and b and c are square-free positive integers.

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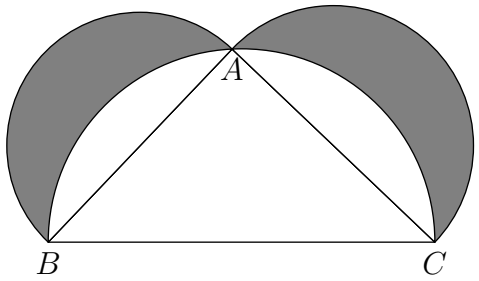
3 The figure below shows a triangle ABC with a semicircle on each of its three sides.



If $AB=20$, $AC=21$, and $BC=29$, what is the area of the shaded region?

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4 If $x>10$, what is the greatest possible value of the expression

$$(\log x)^{\log\log\log x}-(\log\log x)^{\log\log x}?$$

All the logarithms are base 10.

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5 Let $\triangle ABC$ be a triangle with $AB=3$, $BC=4$, and $AC=5$. Let I be the center of the circle inscribed in $\triangle ABC$. What is the product of AI, BI, and CI?

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6 Two circles each have radius 1. No point is inside both circles. The circles are contained in a square. What is the area of the smallest such square?

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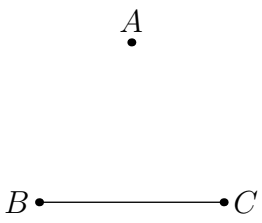
7 If z is a complex number such that

$$z+z^{-1}=\sqrt{3},$$

what is the value of

$$z^{2010}+z^{-2010}\,?$$

8 In the figure below, points A, B, and C are distance 6 from each other. Say that a point X is *reachable* if there is a path (not necessarily straight) connecting A and X of length at most 8 that does not intersect the interior of $\overline{BC}$. (Both X and the path must lie on the plane containing A, B, and C.) Let R be the set of reachable points. What is the area of R?



9 Let ABC be a triangle. Let D be the midpoint of $\overline{BC}$, let E be the midpoint of $\overline{AD}$, and let F be the midpoint of $\overline{BE}$. Let G be the point where the lines AB and CF intersect. What is the value of $\frac{AG}{AB}$?

10 There are real numbers a and b such that for every positive number x, we have the identity

$$\tan^{-1}\left(\frac{1}{x}-\frac{x}{8}\right)+\tan^{-1}(ax)+\tan^{-1}(bx)=\frac{\pi}{2}.$$

(Throughout this equation, $\tan^{-1}$ means the inverse tangent function, sometimes written $\arctan$.) What is the value of a^2+b^2?

11 The sequence $a_0, a_1, a_2, \ldots$ satisfies the recurrence equation

$$a_n = 2a_{n-1} - 2a_{n-2} + a_{n-3}$$

for every integer $n \geq 3$. If $a_{20} = 1$, $a_{25} = 10$, and $a_{30} = 100$, what is the value of a_{1331}?

12 If x is a real number, let $\lfloor x \rfloor$ be the greatest integer that is less than or equal to x. If n is a positive integer, let $S(n)$ be defined by

$$S(n) = \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor + 10\left(n - 10^{\lfloor \log n \rfloor} \cdot \left\lfloor \frac{n}{10^{\lfloor \log n \rfloor}} \right\rfloor\right).$$

(All the logarithms are base 10.) How many integers n from 1 to 2011 (inclusive) satisfy $S(S(n)) = n$?

13 The number 104,060,465 is divisible by a five-digit prime number. What is that prime number?

14 If $0 \le p \le 1$ and $0 \le q \le 1$, define $F(p,q)$ by

$$F(p,q) = -2pq + 3p(1-q) + 3(1-p)q - 4(1-p)(1-q).$$

Define $G(p)$ to be the maximum of $F(p,q)$ over all q (in the interval $0 \le q \le 1$). What is the value of p (in the interval $0 \le p \le 1$) that minimizes $G(p)$?

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15 The game of backgammon has a "doubling" cube, which is like a standard 6-faced die except that its faces are inscribed with the numbers 2, 4, 8, 16, 32, and 64, respectively. After rolling the doubling cube four times at random, we let a be the value of the first roll, b be the value of the second roll, c be the value of the third roll, and d be the value of the fourth roll. What is the probability that $\frac{a+b}{c+d}$ is the average of $\frac{a}{c}$ and $\frac{b}{d}$?

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16 Let N be the number of ordered pairs of integers (x,y) such that

$$4x^2 + 9y^2 \le 1000000000.$$

Let a be the first digit of N (from the left) and let b be the second digit of N. What is the value of $10a + b$?

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17 There is a polynomial P such that for every real number x,

$$x^{512} + x^{256} + 1 = (x^2 + x + 1)P(x).$$

When P is written in standard polynomial form, how many of its coefficients are nonzero?

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18 The polynomial P is a quadratic with integer coefficients. For every positive integer n, the integers $P(n)$ and $P(P(n))$ are relatively prime to n. If $P(3) = 89$, what is the value of $P(10)$?

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19 If $-1 < x < 1$ and $-1 < y < 1$, define the "relativistic sum" $x \oplus y$ to be

$$x \oplus y = \frac{x+y}{1+xy}.$$

The operation $\oplus$ is commutative and associative. Let v be the number

$$v = \frac{\sqrt[7]{17} - 1}{\sqrt[7]{17} + 1}.$$

What is the value of

$$v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \oplus v \,?$$

(In this expression, $\oplus$ appears 13 times.)

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20 Let ABC be an equilateral triangle with each side of length 1. Let X be a point chosen uniformly at random on side $\overline{AB}$. Let Y be a point chosen uniformly at random on side $\overline{AC}$. (Points X and Y are chosen independently.) Let p be the probability that the distance XY is at most $\frac{1}{\sqrt[4]{3}}$. What is the value of $900p$, rounded to the nearest integer?
