Math Prize For Girls Problems 2013

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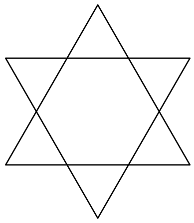
by Ravi B

1 The figure below shows two equilateral triangles each with area 1.



The intersection of the two triangles is a regular hexagon. What is the area of the union of the two triangles?

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2 When the binomial coefficient $\binom{125}{64}$ is written out in base 10, how many zeros are at the rightmost end?

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3 Let S_1, S_2, $\ldots$, S_{125} be 125 sets of 5 numbers each, comprising 625 distinct numbers. Let m_i be the median of S_i. Let M be the median of m_1, m_2, $\ldots$, m_{125}. What is the greatest possible number of the 625 numbers that are less than M?

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4 The MathMatters competition consists of 10 players $P_1, P_2, \ldots, P_{10}$ competing in a ladder-style tournament. Player P_{10} plays a game with P_9: the loser is ranked 10th, while the winner plays P_8. The loser of that game is ranked 9th, while the winner plays P_7. They keep repeating this process until someone plays P_1: the loser of that final game is ranked 2nd, while the winner is ranked 1st. How many different rankings of the players are possible?

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5 Say that a 4-digit positive integer is *mixed* if it has 4 distinct digits, its leftmost digit is neither the biggest nor the smallest of the 4 digits, and its rightmost digit is not the smallest of the 4 digits. For example, 2013 is mixed. How many 4-digit positive integers are mixed?

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6 Three distinct real numbers form (in some order) a 3-term arithmetic sequence, and also form (in possibly a different order) a 3-term geometric sequence. Compute the greatest possible value of the common ratio of this geometric sequence.

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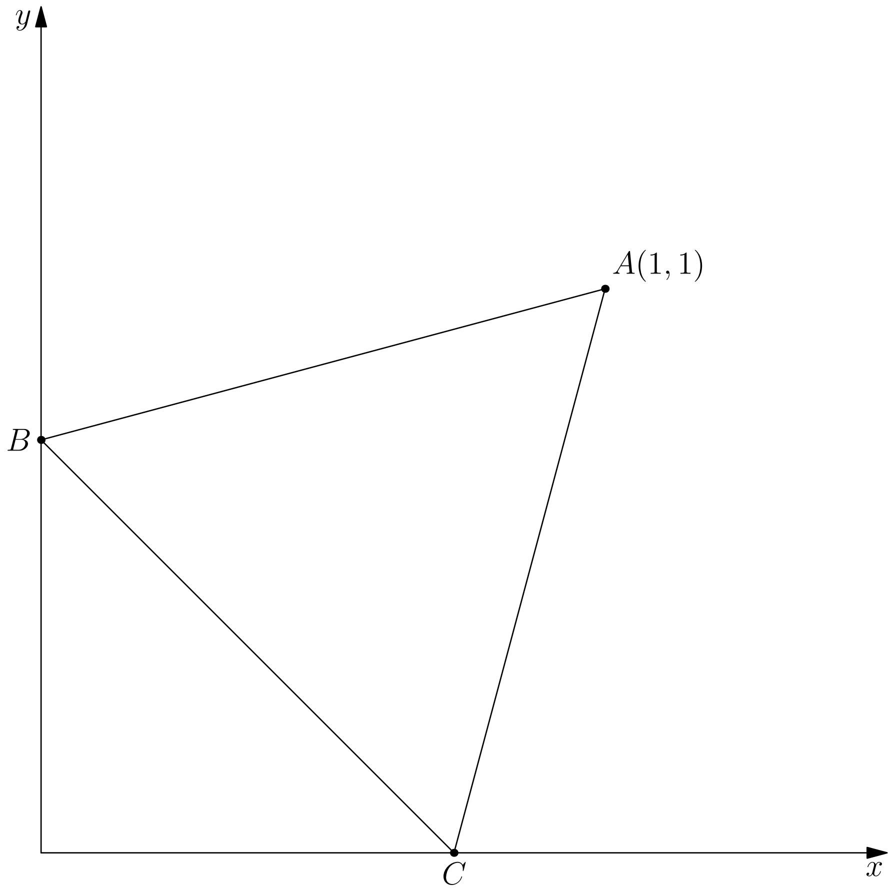
7 In the figure below, $\triangle ABC$ is an equilateral triangle.



Point A has coordinates $(1, 1)$, point B is on the positive y-axis, and point C is on the positive x-axis. What is the area of $\triangle ABC$?

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8 Let R be the set of points (x, y) such that x and y are positive, $x + y$ is at most 2013, and

$$\lceil x \rceil \lfloor y \rfloor = \lfloor x \rfloor \lceil y \rceil.$$

Compute the area of set R. Recall that $\lfloor a \rfloor$ is the greatest integer that is less than or equal to a, and $\lceil a \rceil$ is the least integer that is greater than or equal to a.

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9 Let A and B be distinct positive integers such that each has the same number of positive divisors that 2013 has. Compute the least possible value of $|A - B|$.

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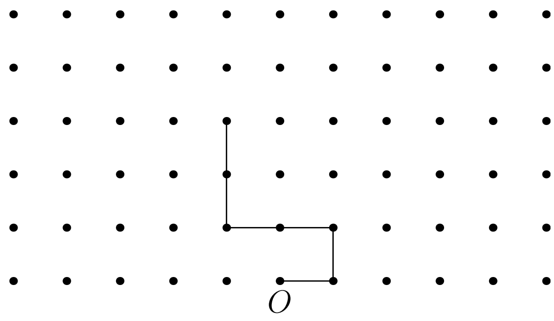
10 The following figure shows a *walk* of length 6:



This walk has three interesting properties:

- It starts at the origin, labelled O.
- Each step is 1 unit north, east, or west. There are no south steps.
- The walk never comes back to a point it has been to.

Let's call a walk with these three properties a *northern walk*. There are 3 northern walks of length 1 and 7 northern walks of length 2. How many northern walks of length 6 are there?

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11 Alice throws two standard dice, with A being the number on her first die and B being the number on her second die. She then draws the line $Ax + By = 2013$. Boris also throws two standard dice, with C being the number on his first die and D being the number on his second die. He then draws the line $Cx + Dy = 2014$. Compute the probability that these two lines are parallel.

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12 The rectangular parallelepiped (box) P has some special properties. If one dimension of P were doubled and another dimension were halved, then the surface area of P would stay the same. If instead one dimension of P were tripled and another dimension were divided by 3, then the surface area of P would still stay the same. If the middle (by length) dimension of P is 1, compute the least possible volume of P.

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13 Each of n boys and n girls chooses a random number from the set $\{1, 2, 3, 4, 5\}$, uniformly and independently. Let p_n be the probability that every boy chooses a different number than every girl. As n approaches infinity, what value does $\sqrt[n]{p_n}$ approach?

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14 How many positive integers n satisfy the inequality

$$\left\lceil \frac{n}{101} \right\rceil + 1 > \frac{n}{100} ?$$

Recall that $\lceil a \rceil$ is the least integer that is greater than or equal to a.

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15 Let $\triangle ABC$ be a triangle with $AB = 7$, $BC = 8$, and $AC = 9$. Point D is on side $\overline{AC}$ such that $\angle CBD$ has measure 45°. What is the length of $\overline{BD}$?

16 If $-3 \le x < \frac{3}{2}$ and $x \neq 1$, define $C(x) = \frac{x^3}{1-x}$. The real root of the cubic $2x^3 + 3x - 7$ is of the form $pC^{-1}(q)$, where p and q are rational numbers. What is the ordered pair (p, q)?

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17 Let f be the function defined by $f(x) = -2\sin(\pi x)$. How many values of x such that $-2 \le x \le 2$ satisfy the equation $f(f(f(x))) = f(x)$?

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18 Ranu starts with one standard die on a table. At each step, she rolls all the dice on the table: if all of them show a 6 on top, then she places one more die on the table; otherwise, she does nothing more on this step. After 2013 such steps, let D be the number of dice on the table. What is the expected value (average value) of 6^D?

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19 If n is a positive integer, let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Compute the value of the infinite sum

$$\sum_{n=1}^{\infty} \frac{\phi(n)2^n}{9^n - 2^n}.$$

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20 Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of real numbers such that $a_0 = \frac{4}{5}$ and

$$a_n = 2a_{n-1}^2 - 1$$

for every positive integer n. Let c be the smallest number such that for every positive integer n, the product of the first n terms satisfies the inequality

$$a_0 a_1 \ldots a_{n-1} \le \frac{c}{2^n}.$$

What is the value of $100c$, rounded to the nearest integer?