Finnish National High School Mathematics Competition 2009

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1 In a plane, the point (x, y) has temperature $x^2 + y^2 - 6x + 4y$. Determine the coldest point of the plane and its temperature.

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2 A polynomial P has integer coefficients and $P(3) = 4$ and $P(4) = 3$. For how many x we might have $P(x) = x$?

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3 The circles $\mathcal{Y}_0$ and $\mathcal{Y}_1$ lies outside each other. Let O_0 be the center of $\mathcal{Y}_0$ and O_1 be the center of $\mathcal{Y}_1$. From O_0, draw the rays which are tangents to $\mathcal{Y}_1$ and similarty from O_1, draw the rays which are tangents to $\mathcal{Y}_0$. Let the intersection points of rays and circle $\mathcal{Y}_i$ be A_i and B_i. Show that the line segments A_0B_0 and A_1B_1 have equal lengths.

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4 We say that the set of step lengths $D \subset \mathbb{Z}_+ = \{1, 2, \ldots\}$ is *excellent* if it has the following property: If we split the set of integers into two subsets A and $\mathbb{Z} \setminus A$, at least other set contains element $a - d, a, a + d$ (i.e. $\{a - d, a, a + d\} \subset A$ or $\{a - d, a, a + d\} \in \mathbb{Z} \setminus A$ from some integer $a \in \mathbb{Z}, d \in D$.) For example the set of one element $\{1\}$ is not excellent as the set of integer can be split into even and odd numbers, and neither of these contains three consecutive integer. Show that the set $\{1, 2, 3, 4\}$ is excellent but it has no proper subset which is excellent.

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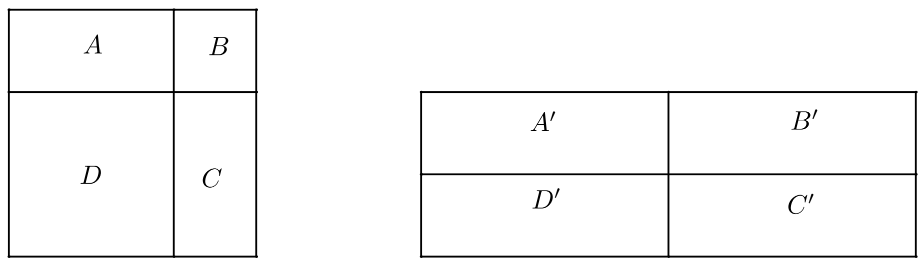
5 As in the picture below, the rectangle on the left hand side has been divided into four parts by line segments which are parallel to a side of the rectangle. The areas of the small rectangles are A, B, C and D. Similarly, the small rectangles on the right hand side have areas A', B', C' and D'. It is known that $A \le A'$, $B \le B'$, $C \le C'$ but $D \le B'$.



Prove that the big rectangle on the left hand side has area smaller or equal to the area of the big rectangle on the right hand side, i.e. $A + B + C + D \le A' + B' + C' + D'$.
