Olympic Revenge 2009

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1 Given a scalene triangle ABC with circuncenter O and circumscribed circle Γ. Let D, E, F the midpoints of BC, AC, AB. Let $M = OE \cap AD$, $N = OF \cap AD$ and $P = CM \cap BN$. Let $X = AO \cap PE$, $Y = AP \cap OF$. Let r the tangent of Γ through A. Prove that r, EF, XY are concurrent.

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2 Prove that $\int_0^{\frac{\pi}{2}} arctg(1-\sin^2 x\cos^2 x)dx = \frac{\pi^2}{4} - \pi arctg\sqrt{\frac{\sqrt{2}-1}{2}}$

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3 Let ABC to be a triangle with incenter I. ω_A, ω_B and ω_C are the incircles of the triangles BIC, CIA and AIB, repectively. After all, T is the tangent point between ω_A and BC. Prove that the other internal common tangent to ω_B and ω_C passes through the point T.

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4 Let $d_i(k)$ the number of divisors of k greater than i.

Let $f(n) = \sum_{i=1}^{\lfloor \frac{n^2}{2} \rfloor} d_i(n^2-i) - 2\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} d_i(n-i)$.
Find all $n \in N$ such that $f(n)$ is a perfect square.

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5 Thin and Fat eat a pizza of $2n$ pieces. Each piece contains a distinct amount of olives between 1 and $2n$. Thin eats the first piece, and the two players alternately eat a piece neighbor of an eaten piece. However, neither Thin nor Fat like olives, so they will choose pieces that minimizes the total amount of olives they eat. For each arrangement σ of the olives, let $s(\sigma)$ the minimal amount of olives that Thin can eat, considering that both play in the best way possible. Let $S(n)$ the maximum of $s(\sigma)$, considering all arrangements. $a)$ Prove that $n^2-1+\lfloor \frac{n}{2} \rfloor \leq S(n) \leq n^2+\lfloor \frac{n}{2} \rfloor$
$b)$ Prove that $S(n) = n^2-1+\frac{n}{2}$ for each even n.

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6 Let $a, n \in \mathbb{Z}_+^*$. a is defined inductively in the base n-*recursive*. We first write a in the base n, e.g., as a sum of terms of the form $k_t n^t$, with $0 \leq k_t < n$. For each exponent t, we write t in the base n-*recursive*, until all the numbers in the representation are less than n. For instance,

$1309 = 3^6 + 2.3^5 + 1.3^4 + 1.3^2 + 1.3 + 1$

$= 3^{2.3} + 2.3^{3+2} + 1.3^{3+1} + 1.3^2 + 1$

Let $x_1 \in \mathbb{Z}$ arbitrary. We define x_n recursively, as following: if $x_{n-1} > 0$, we write x_{n-1} in the base n-*recursive* and we replace all the numbers n for $n+1$ (even the exponents!), so we obtain the successor of x_n. If $x_{n-1} = 0$, then $x_n = 0$.

Example:

$x_1 = 2^{2^2+2+1} + 2^{2+1} + 2 + 1$

$\Rightarrow x_2 = 3^{3^3+3+1} + 3^{3+1} + 3$

$\Rightarrow x_3 = 4^{4^4+4+1} + 4^{4+1} + 3$

$\Rightarrow x_4 = 5^{5^5+5+1} + 5^{5+1} + 2$

$\Rightarrow x_5 = 6^{6^6+6+1} + 6^{6+1} + 1$

$\Rightarrow x_6 = 7^{7^7+7+1} + 7^{7+1}$

$\Rightarrow x_7 = 8^{8^8+8+1} + 7.8^8 + 7.8^7 + 7.8^6 + ... + 7$

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Prove that $\exists N : x_N = 0$.