Olympic Revenge 2010

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by hvaz

1 Prove that the number of ordered triples (x,y,z) such that $(x+y+z)^2 \equiv axyz \mod p$, where $gcd(a,p)=1$ and p is prime is p^2+1.

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2 Joaquim, Jos and Joo participate of the worship of triangle ABC. It is well known that ABC is a random triangle, nothing special. According to the dogmas of the worship, when they form a triangle which is similar to ABC, they will get immortal. Nevertheless, there is a condition: each person must represent a vertice of the triangle. In this case, Joaquim will represent vertice A, Jos vertice B and Joo will represent vertice C. Thus, they must form a triangle which is similar to ABC, in this order.

Suppose all three points are in the Euclidean Plane. Once they are very excited to become immortal, they act in the following way: in each instant t, Joaquim, for example, will move with constant velocity v to the point in the same semi-plan determined by the line which connects the other two points, and which would create a triangle similar to ABC in the desired order. The other participants act in the same way.
If the velocity of all of them is same, and if they initially have a finite, but sufficiently large life, determine if they can get immortal.

Observation: Initially, Joaquim, Jos and Joo do not represent three collinear points in the plane

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3 Prove that there exists a set S of lines in the three dimensional space satisfying the following conditions:

$i)$ For each point P in the space, there exist a unique line of S containing P.

$ii)$ There are no two lines of S which are parallel.

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4 Let a_n and b_n to be two sequences defined as below:

$i)$ $a_1 = 1$

$ii)$ $a_n + b_n = 6n - 1$

$iii)$ a_{n+1} is the least positive integer different of $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$.

Determine a_{2009}.

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5 Secco and Ramon are drunk in the real line over the integer points a and b, respectively. Our real line is a little bit special, though: the interval $(-\infty, 0)$ is covered by a sea of lava. Being aware of this fact, and also because they are drunk, they decided to play the following game: initially they choose an integer number $k > 1$ using a fair dice as large as desired, and therefore they start the game. In the first round, each player writes the point h for which it wants to go.

After that, they throw a coin: if the result is heads, they go to the desired points; otherwise, they go to the points $2g - h$, where g is the point where each of the players were in the precedent round (that is, in the first round $g = a$ for Secco and $g = b$ for Ramon). They repeat this procedure in the other rounds, and the game finishes when some of the player is over a point exactly k times bigger than the other (if both of the player end up in the point 0, the game finishes as well).

Determine, in values of k, the initial values a and b such that Secco and Ramon has a winning strategy to finish the game alive.

Observation: If any of the players fall in the lave, he dies and both of them lose the game

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6 Let ABC to be a triangle and Γ its circumcircle. Also, let D, F, G and E, in this order, on the arc BC which does not contain A satisfying $\angle BAD = \angle CAE$ and $\angle BAF = \angle CAG$. Let $D`, F`, G`$ and $E`$ to be the intersections of AD, AF, AG and AE with BC, respectively. Moreover, X is the intersection of $DF`$ with $EG`$, Y is the intersection of $D`F$ with $E`G$, Z is the intersection of $D`G$ with $E`F$ and W is the intersection of $EF`$ with $DG`$.

Prove that X, Y and A are collinear, such as W, Z and A. Moreover, prove that $\angle BAX = \angle CAZ$.
