Olympic Revenge 2011

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1 Let $p, q, r, s, t \in \mathbb{R}^*_+$ satisfying:
i) $p^2 + pq + q^2 = s^2$
ii) $q^2 + qr + r^2 = t^2$
iii) $r^2 + rp + p^2 = s^2 - st + t^2$
Prove that

$$\frac{s^2 - st + t^2}{s^2t^2} = \frac{r^2}{q^2t^2} + \frac{p^2}{q^2s^2} - \frac{pr}{q^2ts}$$

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2 Let p be a fixed prime. Determine all the integers m, as function of p, such that there exist $a_1, a_2, \ldots, a_p \in \mathbb{Z}$ satisfying

$$m \mid a_1^p + a_2^p + \cdots + a_p^p - (p+1).$$

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3 Let E to be an infinite set of congruent ellipses in the plane, and r a fixed line. It is known that each line parallel to r intersects at least one ellipse belonging to E. Prove that there exist infinitely many triples of ellipses belonging to E, such that there exists a line that intersect the triple of ellipses.

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4 Let $ABCD$ to be a quadrilateral inscribed in a circle Γ. Let r and s to be the tangents to Γ through B and C, respectively, M the intersection between the lines r and AD and N the intersection between the lines s and AD. After all, let E to be the intersection between the lines BN and CM, F the intersection between the lines AE and BC and L the midpoint of BC. Prove that the circuncircle of the triangle DLF is tangent to Γ.

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5 Let $n \in \mathbb{N}$ and $z \in \mathbb{C}^*$. Prove that
$\left| ne^z - \sum_{j=1}^n \left(1 + \frac{z}{j^2}\right)^{j^2} \right| < \frac{1}{3} e^{|z|} \left(\frac{\pi |z|}{2}\right)^2.$
