Problems from Year 28 USAMTS (2016-2017)

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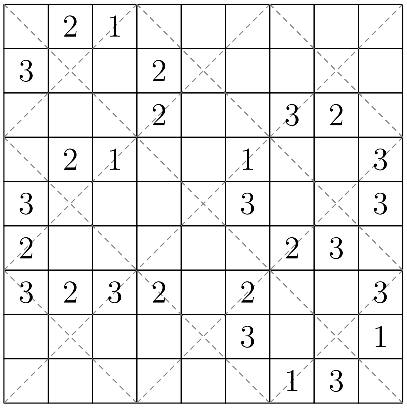
– Round 1 (due 10/17/16)

1: Fill in each cell of the grid with one of the numbers 1, 2, or 3. After all numbers are filled in, if a row, column, or any diagonal has a number of cells equal to a multiple of 3, then it must have the same amount of 1s, 2s, and 3s. (There are 10 such diagonals, and they are all marked in the grid by a gray dashed line.) Some numbers have been given to you.



You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2: A tower of height h is a stack of contiguous rows of squares of height h such that

- the bottom row of the tower has h squares,
- each row above the bottom row has one fewer square than the row below it, and within each row the squares are contiguous,
- the squares in any given row all lie directly above a square in the row below.

A tower is called balanced if when the squares of the tower are colored black and white in a checkerboard fashion, the number of black squares is equal to the number of white squares. For example, the figure above shows a tower of height 5 that is not balanced, since there are 7 white squares and 8 black squares.

How many balanced towers are there of height 2016?

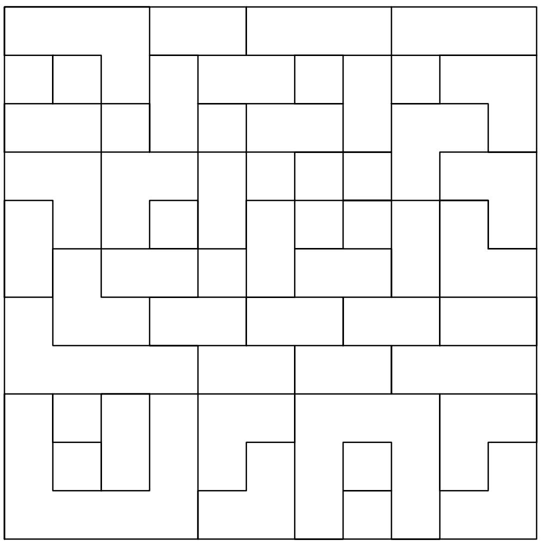
3: Find all positive integers n for which $(x^n + y^n + z^n)/2$ is a perfect square whenever x, y, and z are integers such that $x + y + z = 0$.

4: Find all functions $f(x)$ from nonnegative reals to nonnegative reals such that $f(f(x)) = x^4$ and $f(x) \le Cx^2$ for some constant C.

5: Let $ABCD$ be a convex quadrilateral with perimeter $\frac{5}{2}$ and $AC = BD = 1$. Determine the maximum possible area of $ABCD$.

– Round 2 (due 11/28/16)

1: Shade in some of the regions in the grid to the right so that the shaded area is equal for each of the 11 rows and columns. Regions must be fully shaded or fully unshaded, at least one region must be shaded, and the area of shaded regions must be at most half of the whole grid.



You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2: Find all triples of three-digit positive integers $x < y < z$ with x, y, z in arithmetic progression and $x, y, z + 1000$ in geometric progression.

For this problem, you may use calculators or computers to gain an intuition about how to solve the problem. However, your final submission should include mathematical derivations or proofs and *should not be a solution by exhaustive search.*

3: Suppose m and n are relatively prime positive integers. A regular m-gon and a regular n-gon are inscribed in a circle. Let d be the minimum distance in degrees (of the arc along the circle) between a vertex of the m-gon and a vertex of the n-gon. What is the maximum possible value of d?

4: On Binary Island, residents communicate using special paper. Each piece of paper is a $1 \times n$ row of initially uncolored squares. To send a message, each square on the paper must either be colored either red or green. Unfortunately the paper on the island has become damaged, and each sheet of paper has 10 random consecutive squares each of which is randomly colored red or green.

Malmer and Weven would like to develop a scheme that allows them to send messages of length 2016 between one another. They would like to be able to send any message of length 2016, and they want their scheme to work with perfect accuracy. What is the smallest value of n for which they can develop such a strategy?

Note that when sending a message, one can see which 10 squares are colored and what colors they are. One also knows on which square the message begins, and on which square the message ends.

5: Let $n \geq 4$ and $y_1, \dots, y_n$ real with

$$\sum_{k=1}^{n} y_k = \sum_{k=1}^{n} k y_k = \sum_{k=1}^{n} k^2 y_k = 0$$

and

$$y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = 0$$

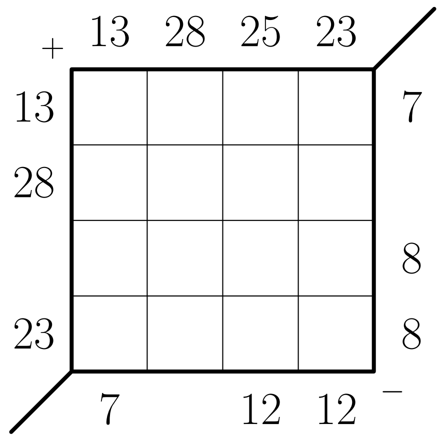
for $1 \leq k \leq n-3$. Prove that

$$\sum_{k=1}^{n} k^3 y_k = 0$$

– Round 3 (due 1/3/17)

1: Fill in each square of the grid with a number from 1 to 16, using each number exactly once. Numbers at the left or top give the largest sum of two numbers in that row or column. Numbers at the right or bottom give the largest difference of two numbers in that row or column. (see the diagram in the first post)

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



2: Malmer Pebane, Fames Jung, and Weven Dare are perfect logicians that always tell the truth. Malmer decides to pose a puzzle to his friends: he tells them that the day of his birthday is at most the number of the month of his birthday. Then Malmer announces that he will whisper the day of his birthday to Fames and the month of his birthday to Weven, and he does exactly that.

After Malmer whispers to both of them, Fames thinks a bit, then says Weven cannot know what Malmers birthday is.

After that, Weven thinks a bit, then says Fames also cannot know what Malmers birthday is.

This exchange repeats, with Fames and Weven speaking alternately and each saying the other cant know Malmers birthday. However, at one point, Weven instead announces Fames and I can now know what Malmers birthday is. Interestingly, that was the longest conversation like that we could have possibly had before both figuring out Malmers birthday.

Find Malmers birthday.

3: An [i]n-city[/i] is an $n \times n$ grid of positive integers such that every entry greater than 1 is the sum of an entry in the same row and an entry in the same column. Shown below is an example 3-city.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 6 & 4 & 1 \end{pmatrix}$$

(a) Construct a 5-city that includes some entry that is at least 150. (It is acceptable simply to write the 5-city. You do not need to explain how you found it.)
(b) Show that for all $n \geq 1$, the largest entry in an n-city is at most $3^{\binom{n}{2}}$.

4: Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be sets of points in the plane. Suppose that for all points x,

$$D(x, A_1) + D(x, A_2) + \cdots + D(x, A_n) \geq D(x, B_1) + D(x, B_2) + \cdots + D(x, B_n)$$

where $D(x, y)$ denotes the distance between x and y. Show that the A_i's and the B_i's share the same center of mass.

5: Consider the set $S = \{q + \frac{1}{q}$, where q ranges over all positive rational numbers$\}$.

(a) Let N be a positive integer. Show that N is the sum of two elements of S if and only if N is the product of two elements of S.

(b) Show that there exist infinitely many positive integers N that cannot be written as the sum of two elements of S.

(c)Show that there exist infinitely many positive integers N that can be written as the sum of two elements of S.
