Olympic Revenge 2012

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1 Let a and b real numbers. Let $f : [a, b] \to \mathbb{R}$ a continuous function. We say that f is "smp" if $[a, b] = [c_0, c_1] \cup [c_1, c_2] ... \cup [c_{n-1}, c_n]$ satisfying $c_0 < c_1 ... < c_n$ and for each $i \in \{0, 1, 2...n-1\}$:
$c_i < x < c_{i+1} \Rightarrow f(c_i) < f(x) < f(c_{i+1})$
or $c_i > x > c_{i+1} \Rightarrow f(c_i) > f(x) > f(c_{i+1})$

Prove that if $f : [a, b] \to \mathbb{R}$ is continuous such that for each $v \in \mathbb{R}$ there are only finitely many x satisfying $f(x) = v$, then f is "smp".

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2 We define $(x_1, x_2, \ldots, x_n)\Delta(y_1, y_2, \ldots, y_n) = (\sum_{i=1}^{n} x_i y_{2-i}, \sum_{i=1}^{n} x_i y_{3-i}, \ldots, \sum_{i=1}^{n} x_i y_{n+1-i})$, where the indices are taken modulo n.

Besides this, if v is a vector, we define $v^k = v$, if $k = 1$, or $v^k = v\Delta v^{k-1}$, otherwise.

Prove that, if $(x_1, x_2, \ldots, x_n)^k = (0, 0, \ldots, 0)$, for some natural number k, then $x_1 = x_2 = \ldots = x_n = 0$.

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3 Let G be a finite graph. Prove that one can partition G into two graphs $A \cup B = G$ such that if we erase all edges conecting a vertex from A to a vertex from B, each vertex of the new graph has even degree.

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4 Say that two sets of positive integers S, T are *k-equivalent* if the sum of the ith powers of elements of S equals the sum of the ith powers of elements of T, for each $i = 1, 2, \ldots, k$. Given k, prove that there are infinitely many numbers N such that $\{1, 2, \ldots, N^{k+1}\}$ can be divided into N subsets, all of which are k-equivalent to each other.

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5 Let $x_1, x_2, \ldots, x_n$ positive real numbers. Prove that:

$$\sum_{cyc} \frac{1}{x_i^3 + x_{i-1}x_i x_{i+1}} \leq \sum_{cyc} \frac{1}{x_i x_{i+1}(x_i + x_{i+1})}$$

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6 Let ABC be an scalene triangle and I and H its incenter, ortocenter respectively.
The incircle touchs BC, CA and AB at D, E an F. DF and AC intersects at K while EF and BC intersets at M.
Shows that KM cannot be paralel to IH.

PS1: The original problem without the adaptation apeared at the Brazilian Olympic Revenge 2011 but it was incorrect.
PS2:The Brazilian Olympic Revenge is a competition for teachers, and the problems are created by the students.

Sorry if I had some English mistakes here.