Olympic Revenge 2014

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1 Let ABC an acute triangle and Γ its circumcircle. The bisector of BAC intersects Γ at $M \neq A$. A line r parallel to BC intersects AC at X and AB at Y. Also, MX and MY intersect Γ again at S and T, respectively.

If XY and ST intersect at P, prove that PA is tangent to Γ.

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2 $a)$ Let n a positive integer. Prove that $gcd(n, \lfloor n\sqrt{2} \rfloor) < \sqrt[4]{8}\sqrt{n}$.

$b)$ Prove that there are infinitely many positive integers n such that $gcd(n, \lfloor n\sqrt{2} \rfloor) > \sqrt[4]{7.99}\sqrt{n}$.

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3 Let n a positive integer. In a $2n \times 2n$ board, $1 \times n$ and $n \times 1$ pieces are arranged without overlap. Call an arrangement **maximal** if it is impossible to put a new piece in the board without overlapping the previous ones.
Find the least k such that there is a **maximal** arrangement that uses k pieces.

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4 Let $a > 1$ be a positive integer and $f \in \mathbb{Z}[x]$ with positive leading coefficient. Let S be the set of integers n such that

$$n \mid a^{f(n)} - 1.$$

Prove that S has density 0; that is, prove that $\lim_{n\to\infty} \frac{|S \cap \{1,\ldots,n\}|}{n} = 0$.
