Uzbekistan National Olympiad 2013

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by mathuz

1 Let real numbers a, b such that $a \geq b \geq 0$. Prove that

$$\sqrt{a^2+b^2} + \sqrt[3]{a^3+b^3} + \sqrt[4]{a^4+b^4} \leq 3a+b.$$

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2 Let x and y are real numbers such that $x^2y^2 + 2yx^2 + 1 = 0$. If $S = \frac{2}{x^2} + 1 + \frac{1}{x} + y(y+2+\frac{1}{x})$, find
(a)max$S$ and
(b) min$S$.

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3 Find all functions $f : Q \to Q$ such that

$$f(x+y) + f(y+z) + f(z+t) + f(t+x) + f(x+z) + f(y+t) \geq 6f(x-3y+5z+7t)$$

for all $x, y, z, t \in Q$.

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4 Let circles Γ and ω are circumcircle and incircle of the triangle ABC, the incircle touches sides BC, CA, AB at the points A_1, B_1, C_1. Let A_2 and B_2 lies the lines A_1I and B_1I (A_1 and A_2 lies different sides from I, B_1 and B_2 lies different sides from I) such that $IA_2 = IB_2 = R$. Prove that :
(a) $AA_2 = BB_2 = IO$;
(b) The lines AA_2 and BB_2 intersect on the circle Γ;

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5 Let $SABC$ is pyramid, such that $SA \leq 4$, $SB \geq 7$, $SC \geq 9$, $AB = 5$, $BC \leq 6$ and $AC \leq 8$. Find max value capacity(volume) of the pyramid $SABC$.
