Uzbekistan National Olympiad 2014

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by ehsan2004, mathuz

1 Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1)$$

is a divisor of $abc - 1$.

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2 Find all functions $f : R \to R$ such that

$$f(x^3) + f(y^3) = (x+y)(f(x^2) + f(y^2) - f(xy))$$

for all $x, y \in R$.

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3 For all $x, y, z \in \mathbb{R}\backslash\{1\}$, such that $xyz = 1$, prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

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4 A circle passes through the points A, C of triangle ABC intersects with the sides AB, BC at points D, E respectively. Let $\frac{BD}{CE} = \frac{3}{2}$, $BE = 4$, $AD = 5$ and $AC = 2\sqrt{7}$.
Find the angle $\angle BDC$.

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5 Let $PA_1A_2...A_{12}$ be the regular pyramid, $A_1A_2...A_{12}$ is regular polygon, S is area of the triangle PA_1A_5 and angle between of the planes $A_1A_2...A_{12}$ and PA_1A_5 is equal to α.
Find the volume of the pyramid.
