Switzerland Team Selection Test 2006

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by BogG

Day 1

1 In the triangle A, B, C, let D be the middle of BC and E the projection of C on AD. Suppose $\angle ACE = \angle ABC$. Show that the triangle ABC is isosceles or rectangle.

2 Let $n \geq 5$ be an integer. Find the biggest integer k such that there always exists a n-gon with exactly k interior right angles. (Find k in terms of n).

3 Let n be natural number. Each of the numbers $\in \{1, 2, \ldots, n\}$ is coloured in black or white. When we choose a number, we flip it's colour and the colour of all the numbers which have at least one common divider with the chosen number. At the beginning all the numbers were coloured white. For which n are all the numbers black after a finite number of changes?

Day 2

1 Let n be natural number and $1 = d_1 < d_2 < \ldots < d_k = n$ be the positive divisors of n.

Find all n such that $2n = d_5^2 + d_6^2 - 1$.

2 Let D be inside $\triangle ABC$ and E on AD different to D. Let ω_1 and ω_2 be the circumscribed circles of $\triangle BDE$ and $\triangle CDE$ respectively. ω_1 and ω_2 intersect BC in the interior points F and G respectively. Let X be the intersection between DG and AB and Y the intersection between DF and AC. Show that XY is $\|$ to BC.

3 Find all the functions $f : \mathbb{R} \to \mathbb{R}$ satisfying for all $x, y \in \mathbb{R}$ $f(f(x) - y^2) = f(x)^2 - 2f(x)y^2 + f(f(y))$.

Day 3

1 The three roots of $P(x) = x^3 - 2x^2 - x + 1$ are $a > b > c \in \mathbb{R}$. Find the value of $a^2b + b^2c + c^2a$. :D

2 We place randomly the numbers $1, 2, \ldots, 2006$ around a circle. A move consists of changing two neighbouring numbers. After a limited numbers of moves all the numbers are diametrically opposite to their starting position. Show that we changed at least once two numbers which had the sum 2007.

3 Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

Day 4

1 Let $a, b, c \in \mathbb{R}^+$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Show $\sqrt{ab+c}+\sqrt{bc+a}+\sqrt{ca+b} \geq \sqrt{abc}+\sqrt{a}+\sqrt{b}+\sqrt{c}$. :D

2 Find all naturals k such that 3^k+5^k is the power of a natural number with the exponent ≥ 2.

3 An airport contains 25 terminals which are two on two connected by tunnels. There is exactly 50 main tunnels which can be traversed in the two directions, the others are with single direction. A group of four terminals is called *good* if of each terminal of the four we can arrive to the 3 others by using only the tunnels connecting them. Find the maximum number of good groups.