www.artofproblemsolving.com/community/c427960
by Tintarn, Snakes, rmtf1111

– Day 1

1 Let the sequence $(a_n)_{n\geqslant 1}$ be defined as:

$$a_n=\sqrt{A_{n+2}^1\sqrt[3]{A_{n+3}^2\sqrt[4]{A_{n+4}^3\sqrt[5]{A_{n+5}^4}}}},$$

where A_m^k are defined by

$$A_m^k=\binom{m}{k}\cdot k!.$$

Prove that

$$a_n<\frac{119}{120}\cdot n+\frac{7}{3}.$$

2 Let

$$f(X)=a_nX^n+a_{n-1}X^{n-1}+\cdots+a_1X+a_0$$

be a polynomial with real coefficients which satisfies

$$a_n\geq a_{n-1}\geq\cdots\geq a_1\geq a_0>0.$$

Prove that for every complex root z of this polynomial, we have $|z|\leq 1$.

3 Let ω be the circumcircle of the acute nonisosceles triangle ΔABC. Point P lies on the altitude from A. Let E and F be the feet of the altitudes from P to CA, BA respectively. Circumcircle of triangle ΔAEF intersects the circle ω in G, different from A. Prove that the lines GP, BE and CF are concurrent.

4 Determine all natural numbers n of the form $n=[a,b]+[b,c]+[c,a]$ where a,b,c are positive integers and $[u,v]$ is the least common multiple of the integers u and v.

– Day 2

5 Find all continuous functions $f:R\rightarrow R$ such, that $f(xy)=f\left(\frac{x^2+y^2}{2}\right)+(x-y)^2$ for any real numbers x and y

6 Let a, b, c be positive real numbers that satisfy $a+b+c=abc$. Prove that

$$\sqrt{(1+a^2)(1+b^2)}+\sqrt{(1+b^2)(1+c^2)}+\sqrt{(1+a^2)(1+c^2)}-\sqrt{(1+a^2)(1+b^2)(1+c^2)} \geq 4.$$

---

7 Let ABC be an acute triangle, and H its orthocenter. The distance from H to rays BC, CA, and AB is denoted by d_a, d_b, and d_c, respectively. Let R be the radius of circumcenter of $\triangle ABC$ and r be the radius of incenter of $\triangle ABC$. Prove the following inequality:

$$d_a+d_b+d_c \leq \frac{3R^2}{4r}$$

.

---

8 At a summer school there are 7 courses. Each participant was a student in at least one course, and each course was taken by exactly 40 students. It is known that for each 2 courses there were at most 9 students who took them both. Prove that at least 120 students participated at this summer school.

– Day 3

---

9 Let

$$P(X)=a_0X^n+a_1X^{n-1}+\cdots+a_n$$

be a polynomial with real coefficients such that $a_0>0$ and

$$a_n \geq a_i \geq 0,$$

for all $i=0,1,2,\ldots,n-1$. Prove that if

$$P^2(X)=b_0X^{2n}+b_1X^{2n-1}+\cdots+b_{n-1}X^{n+1}+\cdots+b_{2n},$$

then $P^2(1) \geq 2b_{n-1}$.

---

10 Let p be an odd prime. Prove that the number

$$\left\lfloor \left(\sqrt{5}+2\right)^p-2^{p+1}\right\rfloor$$

is divisible by $20p$.

---

11 Find all ordered pairs of nonnegative integers (x, y) such that

$$x^4-x^2y^2+y^4+2x^3y-2xy^3=1.$$

---

12 There are 75 points in the plane, no three collinear. Prove that the number of acute triangles is no more than 70% from the total number of triangles with vertices in these points.
