Second Round Olympiad 2004

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by tau172

1 In an acute triangle ABC, point H is the intersection point of altitude CE to AB and altitude BD to AC. A circle with DE as its diameter intersects AB and AC at F and G, respectively. FG and AH intersect at point K. If $BC = 25$, $BD = 20$, and $BE = 7$, find the length of AK.

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2 In a planar rectangular coordinate system, a sequence of points A_n on the positive half of the y-axis and a sequence of points B_n on the curve $y = \sqrt{2x}$ $(x \geq 0)$ satisfy the condition $|OA_n| = |OB_n| = \frac{1}{n}$. The x-intercept of line A_nB_n is a_n, and the x-coordinate of point B_n is b_n, $n \in \mathbb{N}$. Prove that

(1) $a_n > a_{n+1} > 4$, $n \in \mathbb{N}$;

(2) There is $n_0 \in \mathbb{N}$, such that for any $n > n_0$, $\frac{b_2}{b_1} + \frac{b_3}{b_2} + \ldots + \frac{b_n}{b_{n-1}} + \frac{b_{n+1}}{b_n} < n - 2004.$

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3 For integer $n \geq 4$, find the minimal integer $f(n)$, such that for any positive integer m, in any subset with $f(n)$ elements of the set $m, m+1, \ldots, m+n+1$ there are at least 3 relatively prime elements.
