Second Round Olympiad 2006

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– Test 2

1 An ellipse with foci B_0, B_1 intersects AB_i at C_i $(i = 0, 1)$. Let P_0 be a point on ray AB_0. Q_0 is a point on ray C_1B_0 such that $B_0P_0 = B_0Q_0$; P_1 is on ray B_1A such that $C_1Q_0 = C_1P_1$; Q_1 is on ray B_1C_0 such that $B_1P_1 = B_1Q_1$; P_2 is on ray AB_0 such that $C_0Q_1 = C_0Q_2$. Prove that $P_0 = P_2$ and that the four points P_0, Q_0, Q_1, P_1 are concyclic.

2 Let x, y be real numbers. Define a sequence $\{a_n\}$ through the recursive formula

$$a_0 = x, a_1 = y, a_{n+1} = \frac{a_n a_{n-1} + 1}{a_n + a_{n-1}},$$

Find a_n.

3 Solve the system of equations in real numbers:

$$\begin{cases} x - y + z - w = 2 \\ x^2 - y^2 + z^2 - w^2 = 6 \\ x^3 - y^3 + z^3 - w^3 = 20 \\ x^4 - y^4 + z^4 - w^4 = 66 \end{cases}$$

– Test 1

1 Let $\triangle ABC$ be a given triangle. If $|BA - tBC| \geq |AC|$ for any $t \in \mathbb{R}$, then $\triangle ABC$ is **(A)** an acute triangle **(**

2 Suppose $log_x(2x^2 + x - 1) > log_x 2 - 1$. Then the range of x is **(A)** $\frac{1}{2} < x < 1$ **(B)** $x > \frac{1}{2}$and$x \neq 1$ **(C)** $x >$
$x < 1$

3 Suppose $A = x|5x - a \leq 0$, $B = x|6x - b > 0$, $a, b \in \mathbb{N}$, and $A \cap B \cap \mathbb{N} = 2, 3, 4$. The number of such pairs (a, b) is **(A)** 20 **(B)** 25 **(C)** 30 **(D)**42

4 Given a right triangular prism $A_1B_1C_1 - ABC$ with $\angle BAC = \frac{\pi}{2}$ and $AB = AC = AA_1$, let G, E be the midpoints of A_1B_1, CC_1 respectively, and D, F be variable points lying on segments AC, AB (not including endpoints) respectively. If $GD \perp EF$, the range of the length of DF is **(A)** $[\frac{1}{\sqrt{5}}, 1)$ **(B)** $[\frac{1}{5}, 2)$ **(C)** $[1, \sqrt{2})$ **(D)**$[\frac{1}{\sqrt{2}}, \sqrt{2})$

5 Suppose $f(x)=x^3+\log_2(x+\sqrt{x^2+1})$. For any $a,b\in\mathbb{R}$, to satisfy $f(a)+f(b)\geq 0$, the condition $a+b\geq 0$ is

(A) necessary and sufficient **(B)** not necessary but sufficient **(C)** necessary but not sufficient
(D) neither necessary nor sufficient

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6 Let S be the set of all those 2007 place decimal integers $\overline{2s_1a_2a_3\ldots a_{2006}}$ which contain odd number of digit 9 in each sequence $a_1,a_2,a_3,\ldots,a_{2006}$. The cardinal number of S is

(A) $\frac{1}{2}(10^{2006}+8^{2006})$ **(B)** $\frac{1}{2}(10^{2006}-8^{2006})$ **(C)** $10^{2006}+8^{2006}$ **(D)** $10^{2006}-8^{2006}$

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7 Let $f(x)=\sin^4 x-\sin x\cos x+cos^4x$. Find the range of $f(x)$.

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8 Let complex number $z=(a+\cos\theta)+(2a-\sin\theta)i$. Find the range of real number a if $|z|\geq 2$ for any $\theta\in\mathbb{R}$.

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9 Suppose points F_1,F_2 are the left and right foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$ respectively, and point P is on line $l:$, $x-\sqrt{3}y+8+2\sqrt{3}=0$. Find the value of ratio $\frac{|PF_1|}{|PF_2|}$ when $\angle F_1PF_2$ reaches its maximum value.

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10 Suppose four solid iron balls are placed in a cylinder with the radius of 1 cm, such that every two of the four balls are tangent to each other, and the two balls in the lower layer are tangent to the cylinder base. Now put water into the cylinder. Find, in cm^2, the volume of water needed to submerge all the balls.

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11 Find the number of real solutions to the equation $(x^{2006}+1)(1+x^2+x^4+\ldots+x^{2004})=2006x^{2005}$

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12 Suppose there are 8 white balls and 2 red balls in a packet. Each time one ball is drawn and replaced by a white one. Find the probability that the last red ball is drawn in the fourth draw.

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13 Given an integer $n\geq 2$, define $M_0(x_0,y_0)$ to be an intersection point of the parabola $y^2=nx-1$ and the line $y=x$. Prove that for any positive integer m, there exists an integer $k\geq 2$ such that (x_0^m,y_0^m) is an intersection point of $y^2=mx-1$ and the line $y=x$.

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14 Let 2006 be expressed as the sum of five positive integers x_1,x_2,x_3,x_4,x_5, and $S=\sum_{1\leq i<j\leq 5}x_ix_j$.
(A) What value of x_1,x_2,x_3,x_4,x_5 maximizes S? **(A)** Find, with proof, the value of x_1,x_2,x_3,x_4,x_5 which minimizes of S if $|x_i-x_j|\leq 2$ for any $1\leq i,j\leq 5$.

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15 Suppose $f(x)=x^2+a$. Define $f^1(x)=f(x)$, $f^n(x)=f(f^{n-1}(x))$, $n=2,3,\cdots$, and let $M=\{a\in\mathbb{R}||f^n(0)|\leq 2,$ for any $n\in\mathbb{N}\}$. Prove that $M=[-2,\frac{1}{4}]$.
