China Team Selection Test 2015

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TST 1

Day 1

1 The circle Γ through A of triangle ABC meets sides AB, AC at E,F respectively, and circumcircle of ABC at P. Prove: Reflection of P across EF is on BC if and only if Γ passes through O (the circumcentre of ABC).

2 Let $a_1, a_2, a_3, \cdots$ be distinct positive integers, and $0 < c < \frac{3}{2}$. Prove that : There exist infinitely many positive integers k, such that $[a_k, a_{k+1}] > ck$.

3 Fix positive integers k, n. A candy vending machine has many different colours of candy, where there are $2n$ candies of each colour. A couple of kids each buys from the vending machine 2 candies of different colours. Given that for any $k+1$ kids there are two kids who have at least one colour of candy in common, find the maximum number of kids.

Day 2

4 Prove that : For each integer $n \geq 3$, there exists the positive integers $a_1 < a_2 < \cdots < a_n$, such that for $i = 1, 2, \cdots, n-2$, With a_i, a_{i+1}, a_{i+2} may be formed as a triangle side length , and the area of the triangle is a positive integer.

5 FIx positive integer n. Prove: For any positive integers a, b, c not exceeding $3n^2+4n$, there exist integers x, y, z with absolute value not exceeding $2n$ and not all 0, such that $ax+by+cz=0$

6 There are some players in a Ping Pong tournament, where every 2 players play with each other at most once. Given:

(1) Each player wins at least a players, and loses to at least b players. ($a, b \geq 1$)

(2) For any two players A, B, there exist some players $P_1, ..., P_k$ ($k \geq 2$) (where $P_1 = A$,$P_k = B$), such that P_i wins P_{i+1} ($i = 1, 2..., k-1$).

Prove that there exist $a+b+1$ distinct players $Q_1, ...Q_{a+b+1}$, such that Q_i wins Q_{i+1} ($i = 1, ..., a+b$)

TST 2

Day 1

1 For a positive integer n, and a non empty subset A of $\{1, 2, ..., 2n\}$, call A good if the set $\{u \pm v|u, v \in A\}$ does not contain the set $\{1, 2, ..., n\}$. Find the smallest real number c, such that for any positive integer n, and any good subset A of $\{1, 2, ..., 2n\}$, $|A| \leq cn$.

2 Let $a_1, a_2, a_3, \cdots, a_n$ be positive real numbers. For the integers $n \geq 2$, prove that

$$\left(\frac{\sum_{j=1}^{n}\left(\prod_{k=1}^{j} a_k\right)^{\frac{1}{j}}}{\sum_{j=1}^{n} a_j}\right)^{\frac{1}{n}} + \frac{\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}}{\sum_{j=1}^{n}\left(\prod_{k=1}^{j} a_k\right)^{\frac{1}{j}}} \leq \frac{n+1}{n}$$

3 Let $\triangle ABC$ be an acute triangle with circumcenter O and centroid G.
Let D be the midpoint of BC and $E \in \odot(BC)$ be a point inside $\triangle ABC$ such that $AE \perp BC$.
Let $F = EG \cap OD$ and K, L be the point lie on BC such that $FK \parallel OB, FL \parallel OC$.
Let $M \in AB$ be a point such that $MK \perp BC$ and $N \in AC$ be a point such that $NL \perp BC$.
Let ω be a circle tangent to OB, OC at B, C, respectively .

Prove that $\odot(AMN)$ is tangent to ω

Day 2

4 Let n be a positive integer, let $f_1(x), \ldots, f_n(x)$ be n bounded real functions, and let $a_1, \ldots, a_n$ be n distinct reals.
Show that there exists a real number x such that $\sum_{i=1}^n f_i(x) - \sum_{i=1}^n f_i(x - a_i) < 1$.

5 Set S to be a subset of size 68 of $\{1, 2, ..., 2015\}$. Prove that there exist 3 pairwise disjoint, non-empty subsets A, B, C such that $|A| = |B| = |C|$ and $\sum_{a \in A} a = \sum_{b \in B} b = \sum_{c \in C} c$

6 Prove that there exist infinitely many integers n such that $n^2 + 1$ is squarefree.

TST 3

Day 1

1 $\triangle ABC$ is isosceles with $AB = AC > BC$. Let D be a point in its interior such that $DA = DB + DC$. Suppose that the perpendicular bisector of AB meets the external angle bisector of $\angle ADB$ at P, and let Q be the intersection of the perpendicular bisector of AC and the external angle bisector of $\angle ADC$. Prove that B, C, P, Q are concyclic.

2 Let X be a non-empty and finite set, $A_1, ..., A_k$ k subsets of X, satisying:

(1) $|A_i| \leq 3, i = 1, 2, ..., k$
(2) Any element of X is an element of at least 4 sets among $A_1, ..., A_k$.

Show that one can select $[\frac{3k}{7}]$ sets from $A_1, ..., A_k$ such that their union is X.

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3 Let a, b be two integers such that their gcd has at least two prime factors. Let $S = \{x \mid x \in \mathbb{N}, x \equiv a \pmod{b}\}$ and call $y \in S$ irreducible if it cannot be expressed as product of two or more elements of S (not necessarily distinct). Show there exists t such that any element of S can be expressed as product of at most t irreducible elements.

Day 2

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1 Let $x_1, x_2, \cdots, x_n$ $(n \geq 2)$ be a non-decreasing monotonous sequence of positive numbers such that $x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}$ is a non-increasing monotonous sequence .Prove that

$$\frac{\sum_{i=1}^{n} x_i}{n \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}} \leq \frac{n+1}{2\sqrt[n]{n!}}$$

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2 Let G be the complete graph on 2015 vertices. Each edge of G is dyed red, blue or white. For a subset V of vertices of G, and a pair of vertices (u, v), define

$$L(u, v) = \{u, v\} \cup \{w | w \in V \ni \triangle uvw \text{ has exactly 2 red sides}\}$$

Prove that, for any choice of V, there exist at least 120 distinct values of $L(u, v)$.

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3 For all natural numbers n, define $f(n) = \tau(n!) - \tau((n-1)!)$, where $\tau(a)$ denotes the number of positive divisors of a. Prove that there exist infinitely many composite n, such that for all naturals $m < n$, we have $f(m) < f(n)$.
