Second Round Olympiad 2009

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1 Let ω be the circumcircle of acute triangle ABC where $\angle A < \angle B$ and M, N be the midpoints of minor arcs BC, AC of ω respectively. The line PC is parallel to MN, intersecting ω at P (different from C). Let I be the incentre of ABC and let PI intersect ω again at the point T.
1) Prove that $MP \cdot MT = NP \cdot NT$;
2) Let Q be an arbitrary point on minor arc AB and I, J be the incentres of triangles AQC, BCQ. Prove that Q, I, J, T are concyclic.

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2 Let n be a positive integer. Prove that

$$-1 < \sum_{k=1}^{n} \frac{k}{k^2+1} - \ln n \leq \frac{1}{2}$$

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3 Let k, l be two given integers. Prove that there exist infinite many integers $m \geq k$ such that $\gcd\left(\binom{m}{k}, l\right) = 1$.

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4 Let $P = [a_{ij}]_{3\times 9}$ be a 3×9 matrix where $a_{ij} \geq 0$ for all i, j. The following conditions are given:
-Every row consists of distinct numbers;
-$\sum_{i=1}^{3} x_{ij} = 1$ for $1 \leq j \leq 6$;
-$x_{17} = x_{28} = x_{39} = 0$;
-$x_{ij} > 1$ for all $1 \leq i \leq 3$ and $7 \leq j \leq 9$ such that $j - i \neq 6$.
-The first three columns of P satisfy the following property (R): for an arbitrary column $[x_{1k}, x_{2k}, x_{3k}]^T$, $1 \leq k \leq 9$, there exists an $i \in \{1, 2, 3\}$ such that $x_{ik} \leq u_i = \min(x_{i1}, x_{i2}, x_{i3})$.
Prove that:
a) the elements u_1, u_2, u_3 come from three different columns;
b) if a column $[x_{1l}, x_{2l}, x_{3l}]^T$ of P, where $l \geq 4$, satisfies the condition that after replacing the third column of P by it, the first three columns of the newly obtained matrix P' still have property (R), then this column uniquely exists.
