Second Round Olympiad 2011

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–	Second Part
1	Let P,Q be the midpoints of diagonals AC,BD in cyclic quadrilateral $ABCD$. If $\angle BPA = \angle DPA$, prove that $\angle AQB = \angle CQB$.
2	For any integer $n\ge 4$, prove that there exists a n-degree polynomial $f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0$ satisfying the two following properties: **(1)** a_i is a positive integer for any $i=0,1,\ldots,n-1$, and **(2)** For any two positive integers m and k ($k\ge 2$) there exist distinct positive integers $r_1,r_2,...,r_k$, such that $f(m)\ne\prod_{i=1}^{k} f(r_i)$.
3	Given $n\ge 4$ real numbers $a_n>...>a_1>0$. For $r>0$, let $f_n(r)$ be the number of triples (i,j,k) with $1\le i<j<k\le n$ such that $\frac{a_j-a_i}{a_k-a_j}=r$. Prove that $f_n(r)<\frac{n^2}{4}$.
4	Let A be a 3×9 matrix. All elements of A are positive integers. We call an $m\times n$ submatrix of A "ox" if the sum of its elements is divisible by 10, and we call an element of A "carboxylic" if it is not an element of any "ox" submatrix. Find the largest possible number of "carboxylic" elements in A.
–	First Part
1	Let the set $A=(a_1,a_2,a_3,a_4)$. If the sum of elements in every 3-element subset of A makes up the set $B=(-1,5,3,8)$, then find the set A.
2	Find the range of the function $f(x)=\frac{\sqrt{x^2+1}}{x-1}$.
3	Let a,b be positive reals such that $\frac{1}{a}+\frac{1}{b}\le 2\sqrt{2}$ and $(a-b)^2=4(ab)^3$. Find $\log_a b$.
4	If $\cos^5 x-\sin^5 x<7(\sin^3 x-\cos^3 x)$ (for $x\in[0,2\pi)$), then find the range of x.
5	We want to arrange 7 students to attend 5 sports events, but students A and B can't take part in the same event, every event has its own participants, and every student can only attend one event. How many arrangements are there?

6 In a tetrahedral $ABCD$, given that $\angle ADB = \angle BDC = \angle CDA = \frac{\pi}{3}$, $AD = BD = 3$, and $CD = 2$. Find the radius of the circumsphere of $ABCD$.

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7 The line $x - 2y - 1 = 0$ insects the parabola $y^2 = 4x$ at two different points A, B. Let C be a point on the parabola such that $\angle ACB = \frac{\pi}{2}$. Find the coordinate of point C.

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8 Given that $a_n = \binom{200}{n} \cdot 6^{\frac{200-n}{3}} \cdot (\frac{1}{\sqrt{2}})^n$ $(1 \le n \le 95)$. How many integers are there in the sequence $\{a_n\}$?

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9 Let $f(x) = |\log(x+1)|$ and let a, b be two real numbers ($a < b$) satisfying the equations $f(a) = f\left(-\frac{b+1}{a+1}\right)$ and $f\left(10a + 6b + 21\right) = 4\log 2$. Find a, b.

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10 A sequence a_n satisfies $a_1 = 2t - 3$ ($t \ne 1, -1$), and $a_{n+1} = \dfrac{(2t^{n+1} - 3)a_n + 2(t-1)t^n - 1}{a_n + 2t^n - 1}$.

i) Find a_n,

ii) If $t > 0$, compare a_{n+1} with a_n.

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11 A line ℓ with slope of $\frac{1}{3}$ insects the ellipse $C : \frac{x^2}{36} + \frac{y^2}{4} = 1$ at points A, B and the point $P\left(3\sqrt{2}, \sqrt{2}\right)$ is above the line ℓ.

(1) Prove that the locus of the incenter of triangle PAB is a segment,

(2) If $\angle APB = \frac{\pi}{3}$, then find the area of triangle PAB.
