Second Round Olympiad 2012

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– Test 1

1 Let P be a point on the graph of the function $y = x + \frac{2}{x}(x > 0)$. PA, PB are perpendicular to line $y = x$ and $x = 0$, respectively, the feet of perpendicular being A and B. Find the value of $\overrightarrow{PA} \cdot \overrightarrow{PB}$.

2 In $\triangle ABC$, the corresponding sides of angle A, B, C are a, b, c respectively. If $a \cos B - b \cos A = \frac{3}{5}c$, find the value of $\frac{\tan A}{\tan B}$.

3 Suppose that $x, y, z \in [0, 1]$. Find the maximal value of the expression

$$\sqrt{|x-y|} + \sqrt{|y-z|} + \sqrt{|z-x|}.$$

4 Let F be the focus of parabola $y^2 = 2px(p > 0)$, with directrix l and two points A, B on it. Knowing that $\angle AFB = \frac{\pi}{3}$, find the maximal value of $\frac{|MN|}{|AB|}$, where M is the midpoint of AB and N is the projection of M to l.

5 Suppose two regular pyramids with the same base ABC: $P - ABC$ and $Q - ABC$ are circumscribed by the same sphere. If the angle formed by one of the lateral face and the base of pyramid $P - ABC$ is $\frac{\pi}{4}$, find the tangent value of the angle formed by one of the lateral face and the base of the pyramid $Q - ABC$.

6 Let $f(x)$ be an odd function on $\mathbb{R}$, such that $f(x) = x^2$ when $x \geq 0$. Knowing that for all $x \in [a, a+2]$, the inequality $f(x+a) \geq 2f(x)$ holds, find the range of real number a.

7 Find the sum of all integers n satisfying the following inequality:

$$\frac{1}{4} < \sin\frac{\pi}{n} < \frac{1}{3}.$$

8 There are 4 distinct codes used in an intelligence station, one of them applied in each week. No two codes used in two adjacent weeks are the same code. Knowing that code A is used in the first week, find the probability that code A is used in the seventh week.

9 Given a function $f(x)=a\sin x-\frac{1}{2}\cos 2x+a-\frac{3}{a}+\frac{1}{2}$, where $a\in\mathbb{R}, a\neq 0$.
(1) If for any $x\in\mathbb{R}$, inequality $f(x)\le 0$ holds, find all possible value of a.
(2) If $a\ge 2$, and there exists $x\in\mathbb{R}$, such that $f(x)\le 0$. Find all possible value of a.

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10 Given a sequence $\{a_n\}$ whose terms are non-zero real numbers. For any positive integer n, the equality

$$(\sum_{i=1}^{n} a_i)^2=\sum_{i=1}^{n} a_i^3$$

holds.
(1) If $n=3$, find all possible sequence a_1,a_2,a_3;
(2) Does there exist such a sequence $\{a_n\}$ such that $a_{2011}=-2012$?

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11 In the Cartesian plane XOY, there is a rhombus $ABCD$ whose side lengths are all 4 and $|OB|=|OD|=6$, where O is the origin.
(1) Prove that $|OA|\cdot|OB|$ is a constant.
(2) Find the locus of C if A is a point on the semicircle

$$(x-2)^2+y^2=4 \quad (2\le x\le 4).$$

– Test 2

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1 In an acute-angled triangle ABC, $AB>AC$. M,N are distinct points on side BC such that $\angle BAM=\angle CAN$. Let O_1,O_2 be the circumcentres of $\triangle ABC,\triangle AMN$, respectively. Prove that O_1,O_2,A are collinear.

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2 Prove that the set $\{2,2^2,\dots,2^n,\dots\}$ satisfies the following properties:
(1) For every $a\in A, b\in\mathbb{N}$, if $b<2a-1$, then $b(b+1)$ isn't a multiple of $2a$;
(2) For every positive integer $a\notin A, a\neq 1$, there exists a positive integer b, such that $b<2a-1$ and $b(b+1)$ is a multiple of $2a$.

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3 Let $P_0,P_1,P_2,...,P_n$ be $n+1$ points in the plane. Let $d(d>0)$ denote the minimal value of all the distances between any two points. Prove that

$$|P_0P_1|\cdot|P_0P_2|\cdot...\cdot|P_0P_n|>(\frac{d}{3})^n\sqrt{(n+1)!}.$$

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4 Let $S_n=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$, where n is a positive integer. Prove that for any real numbers $a,b,0\le a\le b\le 1$, there exist infinite many $n\in\mathbb{N}$ such that

$$a<S_n-[S_n]<b$$

where $[x]$ represents the largest integer not exceeding x.
