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– Day 1

1. There are 44 distinct holes in a line and 2017 ants. Each ant comes out of a hole and crawls along the line with a constant speed into another hole, then comes in. Let T be the set of moments for which the ant comes in or out of the holes. Given that $|T| \leq 45$ and the speeds of the ants are distinct. Prove that there exists two ants that don't collide.

2. For each positive integer n, set $x_n = \binom{2n}{n}$.
a. Prove that if $\frac{2017^k}{2} < n < 2017^k$ for some positive integer k then 2017 divides x_n.
b. Find all positive integer $h > 1$ such that there exists positive integers N, T such that $(x_n)_{n>N}$ is periodic mod h with period T.

3. Triangle ABC with incircle (I) touches the sides AB, BC, AC at F, D, E, res. I_b, I_c are B- and $C-$ excenters of ABC. P, Q are midpoints of I_bE, I_cF. $(PAC) \cap AB = \{A, R\}$, $(QAB) \cap AC = \{A, S\}$.
a. Prove that PR, QS, AI are concurrent.
b. DE, DF cut I_bI_c at K, J, res. $EJ \cap FK = \{M\}$. PE, QF cut $(PAC), (QAB)$ at X, Y res. Prove that BY, CX, AM are concurrent.

– Day 2

1. Triangle ABC is inscribed in circle (O). A varies on (O) such that $AB > BC$. M is the midpoint of AC. The circle with diameter BM intersects (O) at R. RM intersects (O) at Q and intersects BC at P. The circle with diameter BP intersects AB, BO at K, S in this order.
a. Prove that SR passes through the midpoint of KP.
b. Let N be the midpoint of BC. The radical axis of circles with diameters AN, BM intersects SR at E. Prove that ME always passes through a fixed point.

2. Given 2017 positive real numbers $a_1, a_2, \dots, a_{2017}$. For each $n > 2017$, set

$$a_n = \max\{a_{i_1}a_{i_2}a_{i_3} | i_1 + i_2 + i_3 = n, 1 \leq i_1 \leq i_2 \leq i_3 \leq n-1\}.$$

Prove that there exists a positive integer $m \leq 2017$ and a positive integer $N > 4m$ such that $a_na_{n-4m} = a_{n-2m}^2$ for every $n > N$.

3. For each integer $n > 0$, a permutation $a_1, a_2, \dots, a_{2n}$ of $1, 2, \dots 2n$ is called *beautiful* if for every $1 \leq i < j \leq 2n$, $a_i + a_{n+i} = 2n+1$ and $a_i - a_{i+1} \not\equiv a_j - a_{j+1}$ (mod $2n+1$) (suppose that $a_{2n+1} = a_1$).

a. For $n = 6$, point out a *beautiful* permutation.
b. Prove that there exists a *beautiful* permutation for every n.