Final Round - Korea 2017

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Day 1 25 March 2017

1 A acute triangle $\triangle ABC$ has circumcenter O. The circumcircle of OAB, called O_1, and the circumcircle of OAC, called O_2, meets BC again at $D(\neq B)$ and $E(\neq C)$ respectively. The perpendicular bisector of BC hits AC again at F. Prove that the circumcenter of $\triangle ADE$ lies on AC if and only if the centers of O_1, O_2 and F are colinear.

2 For a positive integer n, $(a_0, a_1, \cdots, a_n)$ is a $n+1$-tuple with integer entries.
For all $k = 0, 1, \cdots, n$, we denote b_k as the number of ks in $(a_0, a_1, \cdots, a_n)$.
For all $k = 0, 1, \cdots, n$, we denote c_k as the number of ks in $(b_0, b_1, \cdots, b_n)$.
Find all $(a_0, a_1, \cdots, a_n)$ which satisfies $a_0 = c_0$, $a_1 = c_1$, $\cdots$, $a_n = c_n$.

3 For a positive integer n, denote $c_n = 2017^n$. A function $f : \mathbb{N} \rightarrow \mathbb{R}$ satisfies the following two conditions.

1. For all positive integers m, n, $f(m+n) \leq 2017 \cdot f(m) \cdot f(n+325)$.

2. For all positive integer n, we have $0 < f(c_{n+1}) < f(c_n)^{2017}$.

Prove that there exists a sequence $a_1, a_2, \cdots$ which satisfies the following.
For all n, k which satisfies $a_k < n$, we have $f(n)^{c_k} < f(c_k)^n$.

Day 2 26 March 2017

4 For a positive integer $n \geq 2$, define a sequence $a_1, a_2, \cdots, a_n$ as the following.

$$a_1 = \frac{n(2n-1)(2n+1)}{3}$$
$$a_k = \frac{(n+k-1)(n-k+1)}{2(k-1)(2k+1)} a_{k-1}, \ (k = 2, 3, \cdots n)$$

(a) Show that $a_1, a_2, \cdots a_n$ are all integers.

(b) Prove that there are exactly one number out of $a_1, a_2, \cdots a_n$ which is not a multiple of $2n-1$ and exactly one number out of $a_1, a_2, \cdots a_n$ which is not a multiple of $2n+1$ if and only if $2n-1$ and $2n+1$ are all primes.

5 Let there be cyclic quadrilateral $ABCD$ with L as the midpoint of AB and M as the midpoint of CD. Let $AC \cap BD = E$, and let rays AB and DC meet again at F. Let $LM \cap DE = P$. Let

Q be the foot of the perpendicular from P to EM. If the orthocenter of $\triangle FLM$ is E, prove the following equality.

$$\frac{EP^2}{EQ} = \frac{1}{2}\left(\frac{BD^2}{DF} - \frac{BC^2}{CF}\right)$$

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6 A room has 2017 boxes in a circle. A set of boxes is *friendly* if there are at least two boxes in the set, and for each boxes in the set, if we go clockwise starting from the box, we would pass either 0 or odd number of boxes before encountering a new box in the set. 30 students enter the room and picks a set of boxes so that the set is friendly, and each students puts a letter inside all of the boxes that he/she chose. If the set of the boxes which have 30 letters inside is not friendly, show that there exists two students A, B and boxes a, b satisfying the following condition.

(i). A chose a but not b, and B chose b but not a.

(ii). Starting from a and going clockwise to b, the number of boxes that we pass through, not including a and b, is not an odd number, and none of A or B chose such boxes that we passed.
