EGMO 2012

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Day 1 April 12th

1 Let ABC be a triangle with circumcentre O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By interior we mean, for example, that the point D lies on the line BC and D is between B and C on that line.)
Let K be the circumcentre of triangle AFE. Prove that the lines DK and BC are perpendicular.

Netherlands (Merlijn Staps)

2 Let n be a positive integer. Find the greatest possible integer m, in terms of n, with the following property: a table with m rows and n columns can be filled with real numbers in such a manner that for any two different rows $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_n]$ the following holds:

$$\max\left(\left|a_1 - b_1\right|, \left|a_2 - b_2\right|, ..., \left|a_n - b_n\right|\right) = 1$$

Poland (Tomasz Kobos)

3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(yf(x+y) + f(x)\right) = 4x + 2yf(x+y)$$

for all $x, y \in \mathbb{R}$.

Netherlands (Birgit van Dalen)

4 A set A of integers is called *sum-full* if $A \subseteq A + A$, i.e. each element $a \in A$ is the sum of some pair of (not necessarily different) elements $b, c \in A$. A set A of integers is said to be *zero-sum-free* if 0 is the only integer that cannot be expressed as the sum of the elements of a finite nonempty subset of A.
Does there exist a sum-full zero-sum-free set of integers?

Romania (Dan Schwarz)

Day 2 April 13th

5 The numbers p and q are prime and satisfy

$$\frac{p}{p+1} + \frac{q+1}{q} = \frac{2n}{n+2}$$

for some positive integer n. Find all possible values of $q - p$.

Luxembourg (Pierre Haas)

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6 There are infinitely many people registered on the social network Mugbook. Some pairs of (different) users are registered as friends, but each person has only finitely many friends. Every user has at least one friend. (Friendship is symmetric; that is, if A is a friend of B, then B is a friend of A.)
Each person is required to designate one of their friends as their best friend. If A designates B as her best friend, then (unfortunately) it does not follow that B necessarily designates A as her best friend. Someone designated as a best friend is called a 1-best friend. More generally, if $n > 1$ is a positive integer, then a user is an n-best friend provided that they have been designated the best friend of someone who is an $(n-1)$-best friend. Someone who is a k-best friend for every positive integer k is called popular.
(a) Prove that every popular person is the best friend of a popular person.
(b) Show that if people can have infinitely many friends, then it is possible that a popular person is not the best friend of a popular person.

Romania (Dan Schwarz)

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7 Let ABC be an acute-angled triangle with circumcircle Γ and orthocentre H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM.
Show that the lines KH, EM and BC are concurrent. (The orthocentre of a triangle is the point on all three of its altitudes.)

Luxembourg (Pierre Haas)

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8 A *word* is a finite sequence of letters from some alphabet. A word is *repetitive* if it is a concatenation of at least two identical subwords (for example, $ababab$ and $abcabc$ are repetitive, but $ababa$ and $aabb$ are not). Prove that if a word has the property that swapping any two adjacent letters makes the word repetitive, then all its letters are identical. (Note that one may swap two adjacent identical letters, leaving a word unchanged.)

Romania (Dan Schwarz)
