Belarusian National Olympiad 2017

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by proximo

– Day 1

– Find all positive real numbers α such that there exists an infinite sequence of positive real numbers $x_1, x_2, \ldots$, such that

$$x_{n+2} = \sqrt{\alpha x_{n+1} - x_n}$$

for all $n \geq 1$

– Let M - be a midpoint of side BC in triangle ABC. A cricumcircle of ABM intersects segment AC at points A and B_1 ($B_1 \neq A$). A circumcircle of AMC intersects segment AB at points A and C_1 ($C_1 \neq A$). Let O be a circumcircle of AC_1B_1. Prove that $OB = OC$

– Let $\overline{a_n \ldots a_1 a_0}$ be a decimal representation of a number 65^k for some $k \geq 2$. Prove that polynomial $a_n x^n + \ldots + a_1 x + a_0$ doesn't have rational roots

– Boris and Eugene are playing the following game : they mark points on the circle in turn. Boris marks the first and paints his point with the white color and Eugene with the black color (no point can be marked twice). As soon as each of them has colored n points any other point on the circle is automatically colored with the color of the nearest marked point (if it doesn't exist, the point remains uncolored). Then Boris and Eugene count the sum of arc length, colored with white and black color respectively. Boy with the greater sum wins.
For all positive integers $n \geq 2$ determine is it possible for one boy to secure his victory. If it's so, then who?

– Day 2

– A cake has a shape of triangle with sides $19, 20$ and 21. It is allowed to cut it it with a line into two pieces and put them on a round plate such that pieces don't overlap each other and don't stick out of the plate. What is the minimal diameter of the plate?

– Let AA_1, BB_1, CC_1 be altitudes of an acute-angeled triangle ABC ($A_1 \in BC, B_1 \in AC, C_1 \in AB$). Let J_a, J_b, J_c be centers of inscribed circles of AC_1B_1, BA_1C_1 and CB_1A_1 respectively. Prove that radius of circumecircle of triangle $J_aJ_bJ_c$ equals radius of inscribed circle of triangle ABC

– Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying the following equation

$$f(x + f(xy)) = xf(1 + f(y))$$

for all positive x and y

– In town N the central square hase a shape of rectangle $n \times m$, composed of squares 1×1. In order, to illuminathe the square, lanterns are placed on the corners of the tiles (including the edge of rectangle), such that every lantern illuminates all tiles in corners of which it is placed. Find the minimal amount of lanterns which can be placed, such that every tile will be illuminated even if one of the lanterns burns out.