Finals 2017

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– Day 1

1 Points P and Q lie respectively on sides AB and AC of a triangle ABC and $BP = CQ$. Segments BQ and CP cross at R. Circumscribed circles of triangles BPR and CQR cross again at point S different from R. Prove that point S lies on the bisector of angle BAC.

2 A sequence $(a_1, a_2, \ldots, a_k)$ consisting of pairwise distinct squares of an $n \times n$ chessboard is called a *cycle* if $k \geq 4$ and squares a_i and a_{i+1} have a common side for all $i = 1, 2, \ldots, k$, where $a_{k+1} = a_1$. Subset X of this chessboard's squares is *mischievous* if each cycle on it contains at least one square in X.

Determine all real numbers C with the following property: for each integer $n \geq 2$, on an $n \times n$ chessboard there exists a mischievous subset consisting of at most Cn^2 squares.

3 Integers $a_1, a_2, \ldots, a_n$ satisfy

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

If m is the number of distinct prime factors of $a_1 a_2 \cdots a_n$, then prove that

$$(a_1 a_2 \cdots a_n)^{m-1} \geq (n!)^m.$$

– Day 2

4 Prove that the set of positive integers $\mathbb{Z}^+$ can be represented as a sum of five pairwise distinct subsets with the following property: each 5-tuple of numbers of form $(n, 2n, 3n, 4n, 5n)$, where $n \in \mathbb{Z}^+$, contains exactly one number from each of these five subsets.

5 Point M is the midpoint of BC of a triangle ABC, in which $AB = AC$. Point D is the orthogonal projection of M on AB. Circle ω is inscribed in triangle ACD and tangent to segments AD and AC at K and L respectively. Lines tangent to ω which pass through M cross line KL at X and Y, where points X, K, L and Y lie on KL in this specific order. Prove that points M, D, X and Y are concyclic.

6 Three sequences $(a_0, a_1, \ldots, a_n)$, $(b_0, b_1, \ldots, b_n)$, $(c_0, c_1, \ldots, c_{2n})$ of non-negative real numbers

are given such that for all $0 \le i, j \le n$ we have $a_i b_j \le (c_{i+j})^2$. Prove that

$$\sum_{i=0}^{n} a_i \cdot \sum_{j=0}^{n} b_j \le \left(\sum_{k=0}^{2n} c_k \right)^2 .$$