ELMO Problems 2009

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by v_Enhance

Day 1

1 Let a, b, c be positive integers such that $a^2 - bc$ is a square. Prove that $2a + b + c$ is not prime.

Evan o'Dorney

2 Let ABC be a triangle such that $AB < AC$. Let P lie on a line through A parallel to line BC such that C and P are on the same side of line AB. Let M be the midpoint of segment BC. Define D on segment BC such that $\angle BAD = \angle CAM$, and define T on the extension of ray CB beyond B so that $\angle BAT = \angle CAP$. Given that lines PC and AD intersect at Q, that lines PD and AB intersect at R, and that S is the midpoint of segment DT, prove that if A,P,Q, and R lie on a circle, then Q, R, and S are collinear.

David Rush

3 Let a, b, c be nonnegative real numbers. Prove that

$$a(a-b)(a-2b) + b(b-c)(b-2c) + c(c-a)(c-2a) \geq 0.$$

Wenyu Cao

Day 2

4 Let n be a positive integer. Given n^2 points in a unit square, prove that there exists a broken line of length $2n + 1$ that passes through all the points.

Allen Yuan

5 Let $ABCDEFG$ be a regular heptagon with center O. Let M be the centroid of $\triangle ABD$. Prove that $\cos^2(\angle GOM)$ is rational and determine its value.

Evan o'Dorney

6 Let p be an odd prime and x be an integer such that $p \mid x^3 - 1$ but $p \nmid x - 1$. Prove that

$$p \mid (p-1)! \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \frac{x^{p-1}}{p-1} \right).$$

John Berman