ELMO Problems 2011

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by math154

Day 1 June 18th

1 Let $ABCD$ be a convex quadrilateral. Let E, F, G, H be points on segments AB, BC, CD, DA, respectively, and let P be the intersection of EG and FH. Given that quadrilaterals $HAEP$, $EBFP$, $FCGP$, $GDHP$ all have inscribed circles, prove that $ABCD$ also has an inscribed circle.

Evan O'Dorney.

2 Wanda the Worm likes to eat Pascal's triangle. One day, she starts at the top of the triangle and eats $\binom{0}{0} = 1$. Each move, she travels to an adjacent positive integer and eats it, but she can never return to a spot that she has previously eaten. If Wanda can never eat numbers a, b, c such that $a + b = c$, prove that it is possible for her to eat 100,000 numbers in the first 2011 rows given that she is not restricted to traveling only in the first 2011 rows.

(Here, the $n + 1$st row of Pascal's triangle consists of entries of the form $\binom{n}{k}$ for integers $0 \leq k \leq n$. Thus, the entry $\binom{n}{k}$ is considered adjacent to the entries $\binom{n-1}{k-1}$, $\binom{n-1}{k}$, $\binom{n}{k-1}$, $\binom{n}{k+1}$, $\binom{n+1}{k}$, $\binom{n+1}{k+1}$.)

Linus Hamilton.

3 Determine whether there exist two reals x, y and a sequence $\{a_n\}_{n=0}^{\infty}$ of nonzero reals such that $a_{n+2} = xa_{n+1} + ya_n$ for all $n \geq 0$ and for every positive real number r, there exist positive integers i, j such that $|a_i| < r < |a_j|$.

Alex Zhu.

Day 2 June 19th

4 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that whenever $a > b > c > d > 0$ and $ad = bc$,

$$f(a + d) + f(b - c) = f(a - d) + f(b + c).$$

Calvin Deng.

5 Let $p > 13$ be a prime of the form $2q + 1$, where q is prime. Find the number of ordered pairs of integers (m, n) such that $0 \leq m < n < p - 1$ and

$$3^m + (-12)^m \equiv 3^n + (-12)^n \pmod{p}.$$

Alex Zhu.

The original version asked for the number of solutions to $2^m + 3^n \equiv 2^n + 3^n \pmod{p}$ (still $0 \le m < n < p-1$), where p is a Fermat prime.

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6 Consider the infinite grid of lattice points in $\mathbb{Z}^3$. Little D and Big Z play a game, where Little D first loses a shoe on an unmunched point in the grid. Then, Big Z munches a shoe-free plane perpendicular to one of the coordinate axes. They continue to alternate turns in this fashion, with Little D's goal to lose a shoe on each of n consecutive lattice points on a line parallel to one of the coordinate axes. Determine all n for which Little D can accomplish his goal.

David Yang.
