ELMO Problems 2012

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by math154

Day 1 June 16th

1 In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

Ray Li.

2 Find all ordered pairs of positive integers (m, n) for which there exists a set $C = \{c_1, \ldots, c_k\}$ ($k \geq 1$) of colors and an assignment of colors to each of the mn unit squares of a $m \times n$ grid such that for every color $c_i \in C$ and unit square S of color c_i, exactly two direct (non-diagonal) neighbors of S have color c_i.

David Yang.

3 Let f, g be polynomials with complex coefficients such that $\gcd(\deg f, \deg g) = 1$. Suppose that there exist polynomials $P(x, y)$ and $Q(x, y)$ with complex coefficients such that $f(x) + g(y) = P(x, y)Q(x, y)$. Show that one of P and Q must be constant.

Victor Wang.

Day 2 June 17th

4 Let a_0, b_0 be positive integers, and define $a_{i+1} = a_i + \lfloor \sqrt{b_i} \rfloor$ and $b_{i+1} = b_i + \lfloor \sqrt{a_i} \rfloor$ for all $i \geq 0$. Show that there exists a positive integer n such that $a_n = b_n$.

David Yang.

5 Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

Calvin Deng.

6 A diabolical combination lock has n dials (each with c possible states), where $n, c > 1$. The dials are initially set to states $d_1, d_2, \ldots, d_n$, where $0 \leq d_i \leq c - 1$ for each $1 \leq i \leq n$. Unfortunately, the actual states of the dials (the d_i's) are concealed, and the initial settings of the dials are also unknown. On a given turn, one may advance each dial by an integer amount c_i ($0 \leq c_i \leq c - 1$), so that every dial is now in a state $d_i' \equiv d_i + c_i \pmod{c}$ with $0 \leq d_i' \leq c - 1$. After each turn, the lock opens if and only if all of the dials are set to the zero state; otherwise, the lock selects a random integer k and cyclically shifts the d_i's by k (so that for every i, d_i is replaced by d_{i-k}, where indices are taken modulo n).

Show that the lock can always be opened, regardless of the choices of the initial configuration and the choices of k (which may vary from turn to turn), if and only if n and c are powers of the same prime.

Bobby Shen.