ELMO Problems 2014

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by v_Enhance

Day 1 June 15th

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1 Find all triples (f,g,h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Proposed by Evan Chen

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2 Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3,4,5,6\}$ and n is a positive integer.
Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Proposed by Matthew Babbitt

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3 We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \geq 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Proposed by Sammy Luo

Day 2 June 21st

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4 Let n be a positive integer and let $a_1, a_2, \dots, a_n$ be real numbers strictly between 0 and 1. For any subset S of $\{1,2,...,n\}$, define

$$f(S) = \prod_{i \in S} a_i \cdot \prod_{j \notin S} (1-a_j).$$

Suppose that $\sum_{|S| \text{ odd}} f(S) = \frac{1}{2}$. Prove that $a_k = \frac{1}{2}$ for some k. (Here the sum ranges over all subsets of $\{1, 2, ..., n\}$ with an odd number of elements.)

Proposed by Kevin Sun

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5 Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumcircles of triangles BOC and BHC, respectively. Suppose the circle with diameter $\overline{AO}$ intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the circle with diameter $\overline{AH}$ intersects ω_2 again at N, and line AN intersects ω_2 again at Y. Prove that lines MN and XY are parallel.

Proposed by Sammy Luo

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6 A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares s_1, s_2, $\dots$, s_n such that s_i and s_{i+1} share an edge for $i = 1, 2, \dots, n$ (here $s_{n+1} = s_1$).

What is the maximum possible number of filled black squares?

Proposed by David Yang
