Macedonian National Olympiad 2017

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Problem 1 Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that for each natural integer $n>1$ and for all $x,y\in\mathbb{N}$ the following holds:

$$f(x+y)=f(x)+f(y)+\sum_{k=1}^{n-1}\binom{n}{k}x^{n-k}y^k$$

Problem 2 Find all natural integers n such that $(n^3+39n-2)n!+17\cdot 21^n+5$ is a square.

Problem 3 Let $x,y,z\in\mathbb{R}$ such that $xyz=1$. Prove that:

$$\left(x^4+\frac{z^2}{y^2}\right)\left(y^4+\frac{x^2}{z^2}\right)\left(z^4+\frac{y^2}{x^2}\right)\geq\left(\frac{x^2}{y}+1\right)\left(\frac{y^2}{z}+1\right)\left(\frac{z^2}{x}+1\right).$$

Problem 4 Let O be the circumcenter of the acute triangle ABC ($AB<AC$). Let A_1 and P be the feet of the perpendicular lines drawn from A and O to BC, respectively. The lines BO and CO intersect AA_1 in D and E, respectively. Let F be the second intersection point of $\odot ABD$ and $\odot ACE$. Prove that the angle bisector od $\angle FAP$ passes through the incenter of $\triangle ABC$.

Problem 5 Let $n>1\in\mathbb{N}$ and $a_1,a_2,...,a_n$ be a sequence of n natural integers. Let:

$$b_1=\left[\frac{a_2+\cdots+a_n}{n-1}\right],b_i=\left[\frac{a_1+\cdots+a_{i-1}+a_{i+1}+\cdots+a_n}{n-1}\right],b_n=\left[\frac{a_1+\cdots+a_{n-1}}{n-1}\right]$$

Define a mapping f by $f(a_1,a_2,\cdots a_n)=(b_1,b_2,\cdots,b_n)$.

a) Let $g:\mathbb{N}\to\mathbb{N}$ be a function such that $g(1)$ is the number of different elements in $f(a_1,a_2,\cdots a_n)$ and $g(m)$ is the number od different elements in $f^m(a_1,a_2,\cdots a_n)=f(f^{m-1}(a_1,a_2,\cdots a_n));m>1$. Prove that $\exists k_0\in\mathbb{N}$ s.t. for $m\geq k_0$ the function $g(m)$ is periodic.

b) Prove that $\sum_{m=1}^{k}\frac{g(m)}{m(m+1)}<C$ for all $k\in\mathbb{N}$, where C is a function that doesn't depend on k.
