IMC 1997

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Day 1

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1 Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive reals with $\lim_{n\rightarrow+\infty}\epsilon_n = 0$. Find

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\epsilon_n\right)$$

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2 Let a_n be a sequence of reals. Suppose $\sum a_n$ converges. Do these sums converge aswell?

(a) $a_1 + a_2 + (a_4 + a_3) + (a_8 + ... + a_5) + (a_{16} + ... + a_9) + ...$

(b) $a_1 + a_2 + (a_3) + (a_4) + (a_5 + a_7) + (a_6 + a_8) + (a_9 + a_{11} + a_{13} + a_{15}) + (a_{10} + a_{12} + a_{14} + a_{16}) + (a_{17} +$

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3 Let $A, B \in \mathbb{R}^{n\times n}$ with $A^2 + B^2 = AB$. Prove that if $BA - AB$ is invertible then $3|n$.

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4 Let α be a real number, $1 < \alpha < 2$.

(a) Show that α can uniquely be represented as the infinte product

$$\alpha = \left(1+\frac{1}{n_1}\right)\left(1+\frac{1}{n_2}\right)\cdots$$

with n_i positive integers satisfying $n_i^2 \le n_{i+1}$.

(b) Show that $\alpha \in \mathbb{Q}$ iff from some k onwards we have $n_{k+1} = n_k^2$.

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5 For postive integer n consider the hyperplane

$$R_0^n = x = (x_1x_2...x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0$$

and the lattice

$$Z_0^n = \{y \in R_0^n : \ (\forall i : y_i \in \mathbb{N})\}$$

Define the quasi-norm in $\mathbb{R}^n$ by $\|x\|_p = \sqrt[p]{\sum_{i=1}^{n}|x_i|^p}$ if $0 < p < \infty$ and $\|x\|_\infty = \max_i |x_i|$.

(a) If $x \in R_0^n$ so that $\max x_i - \min x_i \leq 1$ then prove that $\forall p \in [1,\infty], \forall y \in Z_0^n$ we have $\|x\|_p \leq \|x+y\|_p$
(b) Prove that for every $p \in]0,1[$, there exist $n \in \mathbb{N}, x \in R_0^n, y \in Z_0^n$ with $\max x_i - \min x_i \leq 1$ and $\|x\|_p > \|x+y\|_p$

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6 Suppose F is a family of finite subsets of $\mathbb{N}$ and for any 2 sets $A, B \in F$ we have $A \cap B \neq \emptyset$.

(a) Is it true that there is a finite subset Y of $\mathbb{N}$ such that for any $A, B \in F$ we have $A \cap B \cap Y \neq \emptyset$?
(b) Is the above true if we assume that all members of F have the same size?

Day 2

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1 Let $f \in C^3(\mathbb{R})$ nonnegative function with $f(0) = f'(0) = 0, f''(0) > 0$. Define $g(x)$ as follows:

$$\left\{ \begin{array}{ll} g(x) = (\frac{\sqrt{f(x)}}{f'(x)})' & \text{for} \quad x \neq 0 \\ g(x) = 0 & \text{for} \quad x = 0 \end{array} \right.$$

(a) Show that g is bounded in some neighbourhood of 0.
(b) Is the above true for $f \in C^2(\mathbb{R})$?

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2 Let $M \in GL_{2n}(K)$, represented in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

Show that $\det M . \det H = \det A$.

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3 Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(\log n)}{n^{\alpha}}$ converges iff $\alpha > 0$.

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4 (a) Let $f : \mathbb{R}^{n \times n} \to \mathbb{R}$ be a linear mapping. Prove that $\exists ! C \in \mathbb{R}^{n \times n}$ such that $f(A) = Tr(AC), \forall A \in \mathbb{R}^{n \times n}$.

(b) Suppose in addtion that $\forall A, B \in \mathbb{R}^{n \times n} : f(AB) = f(BA)$. Prove that $\exists \lambda \in \mathbb{R} : f(A) = \lambda Tr(A)$

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5 Let X be an arbitrary set and f a bijection from X to X. Show that there exist bijections $g,\ g' : X \to X$ s.t. $f = g \circ g',\ g \circ g = g' \circ g' = 1_X$.

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6 Let $f : [0,1] \to \mathbb{R}$ continuous. We say that f crosses the axis at x if $f(x) = 0$ but $\exists y, z \in [x-\epsilon, x+\epsilon] : f(y) < 0 < f(z)$ for any ϵ.

(a) Give an example of a function that crosses the axis infinitely often.

(b) Can a continuous function cross the axis uncountably often?