IMC 1998

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–	Day 1
1	Let V be a 10-dimensional real vector space and U_1, U_2 two linear subspaces such that $U_1 \subseteq U_2, \dim U_1 = 3, \dim U_2 = 6$. Let ε be the set of all linear maps $T : V \rightarrow V$ which have $T(U_1) \subseteq U_1, T(U_2) \subseteq U_2$. Calculate the dimension of ε. (again, all as real vector spaces)
2	Consider the following statement: for any permutation $\pi_1 \neq \mathbb{I}$ of $\{1, 2, ..., n\}$ there is a permutation π_2 such that any permutation on these numbers can be obtained by a finite compostion of π_1 and π_2. (a) Prove the statement for $n = 3$ and $n = 5$. (b) Disprove the statement for $n = 4$.
3	Let $f(x) = 2x(1-x), x \in \mathbb{R}$ and denote $f_n = f \circ f \circ ... \circ f$, n times. (a) Find $\lim_{n\rightarrow\infty} \int_0^1 f_n(x)dx$. (b) Now compute $\int_0^1 f_n(x)dx$.
4	The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is twice differentiable and satisfies $f(0) = 2, f'(0) = -2, f(1) = 1$. Prove that there is a $\xi \in]0, 1[$ for which we have $f(\xi) \cdot f'(\xi) + f''(\xi) = 0$.
5	Let P be a polynomial of degree n with only real zeros and real coefficients. Prove that for every real x we have $(n-1)(P'(x))^2 \geq nP(x)P''(x)$. When does equality occur?
6	Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function satisfying $xf(y) + yf(x) \leq 1$ for every $x, y \in [0, 1]$. (a) Show that $\int_0^1 f(x)dx \leq \frac{\pi}{4}$. (b) Find such a funtion for which equality occurs.
–	Day 2
1	V is a real vector space and $f, f_i : V \rightarrow \mathbb{R}$ are linear for $i = 1, 2, ..., k$. Also f is zero at all points for which all of f_i are zero. Show that f is a linear combination of the f_i.
2	S ist the set of all cubic polynomials f with $\lvert f(\pm 1)\rvert \leq 1$ and $\lvert f(\pm \frac{1}{2})\rvert \leq 1$. Find $\sup_{f\in S} \max_{-1\leq x\leq 1} \lvert f''(x)\rvert$ and all members of f which give equality.

3 Given $0 < c < 1$, we define $f(x) = \begin{cases} \frac{x}{c} & x \in [0, c] \\ \frac{1-x}{1-c} & x \in [c, 1] \end{cases}$
Let $f^n(x) = f(f(...f(x)))$. Show that for each positive integer n, f^n has a non-zero finite nunber of fixed points which aren't fixed points of f^k for $k < n$.

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4 Let $S_n = \{1, 2, ..., n\}$. How many functions $f : S_n \to S_n$ satisfy $f(k) \leq f(k+1)$ and $f(k) = f(f(k+1))$ for $k < n$?

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5 S is a family of balls in $\mathbb{R}^n$ ($n > 1$) such that the intersection of any two contains at most one point. Show that the set of points belonging to at least two members of S is countable.

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6 $f : (0, 1) \to [0, \infty)$ is zero except at a countable set of points $a_1, a_2, a_3, ...$. Let $b_n = f(a_n)$. Show that if $\sum b_n$ converges, then f is differentiable at at least one point. Show that for any sequence b_n of non-negative reals with $\sum b_n = \infty$, we can find a sequence a_n such that the function f defined as above is nowhere differentiable.
