IMC 1999

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Day 1

1 a) Show that $\forall n \in \mathbb{N}_0, \exists A \in \mathbb{R}^{n\times n} : A^3 = A + I$.
b) Show that $\det(A) > 0, \forall A$ fulfilling the above condition.

2 Does there exist a bijective map $f : \mathbb{N} \to \mathbb{N}$ so that $\sum_{n=1}^{\infty} \frac{f(n)}{n^2}$ is finite?

3 Suppose that $f : \mathbb{R} \to \mathbb{R}$ fulfils $\left|\sum_{k=1}^{n} 3^k \left(f(x+ky) - f(x-ky)\right)\right| \leq 1$ for all $n \in \mathbb{N}, x, y \in \mathbb{R}$. Prove that f is a constant function.

4 Find all strictly monotonic functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ for which $f\left(\frac{x^2}{f(x)}\right) = x$ for all x.

5 Suppose that $2n$ points of an $n \times n$ grid are marked. Show that for some $k > 1$ one can select $2k$ distinct marked points, say $a_1, ..., a_{2k}$, such that a_{2i-1} and a_{2i} are in the same row, a_{2i} and a_{2i+1} are in the same column, $\forall i$, indices taken mod 2n.

6 (a) Let $p > 1$ a real number. Find a real constant c_p for which the following statement holds: If $f : [-1, 1] \to \mathbb{R}$ is a continuously differentiable function with $f(1) > f(-1)$ and $|f'(y)| \leq 1 \forall y \in [-1, 1]$, then $\exists x \in [-1, 1] : f'(x) > 0$ so that $\forall y \in [-1, 1] : |f(y) - f(x)| \leq c_p \sqrt[p]{f'(x)}|y - x|$.

(b) What if $p = 1$?

Day 2

1 Let R be a ring where $\forall a \in R : a^2 = 0$. Prove that $abc + abc = 0$ for all $a, b, c \in R$.

2 We roll a regular 6-sided dice n times. What is the probabilty that the total number of eyes rolled is a multiple of 5?

3 Let $x_i \geq -1$ and $\sum_{i=1}^{n} x_i^3 = 0$. Prove $\sum_{i=1}^{n} x_i \leq \frac{n}{3}$.

4 Prove that there's no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x)^2 \geq f(x+y)\left(f(x) + y\right)$ for all $x, y > 0$.

5 Let S be the set of words made from the letters a, b and c. The equivalence relation $\sim$ on S satisfies

$$uu \sim u$$

$$u \sim v \Rightarrow uw \sim vw \text{ and } wu \sim wv$$

for all words u, v and w. Prove that every word in S is equivalent to a word of length ≤ 8.

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6 Let A be a subset of $\mathbb{Z}/n\mathbb{Z}$ with at most $\frac{\ln(n)}{100}$ elements.

Define $f(r) = \sum_{s \in A} e^{\dfrac{2\pi i r s}{n}}$. Show that for some $r \neq 0$ we have $|f(r)| \geq \frac{|A|}{2}$.
