AoPS Online
Art of Problem Solving

IMC 2000

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by Fermat -Euler, alekk, Peter, Valentin Vornicu, sqrtX

Day 1

1 Does every monotone increasing function $f:[0,1]\rightarrow[0,1]$ have a fixed point? What about every monotone decreasing function?

2 Let $p(x)=x^5+x$ and $q(x)=x^5+x^2$, Find al pairs $(w,z)\in\mathbb{C}\times\mathbb{C}$, $w\neq z$ for which $p(w)=p(z), q(w)=q(z)$.

3 Let $A,B\in\mathbb{C}^{n\times n}$ with $\rho(AB-BA)=1$. Show that $(AB-BA)^2=0$.

4 Let (x_i) be a decreasing sequence of positive reals, then show that:

(a) for every positive integer n we have $\sqrt{\sum_{i=1}^n x_i^2}\leq\sum_{i=1}^n\frac{x_i}{\sqrt{i}}$.

(b) there is a constant C for which we have $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\sqrt{\sum_{i=k}^{\infty}x_i^2}\leq C\sum_{i=1}^{\infty}x_i$.

5 Let R be a ring of characteristic zero. Let $e,f,g\in R$ be idempotent elements (an element x is called idempotent if $x^2=x$) satisfying $e+f+g=0$. Show that $e=f=g=0$.

6 Let $f:\mathbb{R}\rightarrow]0,+\infty[$ be an increasing differentiable function with $\lim_{x\rightarrow+\infty}f(x)=+\infty$ and f' is bounded, and let $F(x)=\int_0^x f(t)dt$.
Define the sequence (a_n) recursively by $a_0=1, a_{n+1}=a_n+\frac{1}{f(a_n)}$
Define the sequence (b_n) by $b_n=F^{-1}(n)$.
Prove that $\lim_{x\rightarrow+\infty}(a_n-b_n)=0$.

Day 2

1 Show that a square may be partitioned into n smaller squares for sufficiently large n.
Show that for some constant $N(d)$, a d-dimensional cube can be partitioned into n smaller cubes if $n\geq N(d)$.

2 Let f be continuous and nowhere monotone on $[0,1]$. Show that the set of points on which f obtains a local minimum is dense.

3 Let $p(z)$ be a polynomial of degree $n>0$ with complex coefficients. Prove that there are at least $n+1$ complex numbers z for which $p(z)\in\{0,1\}$.

4 Let $OABC$ be a tetrahedon with $\angle BOC = \alpha, \angle COA = \beta$ and $\angle AOB = \gamma$. The angle between the faces OAB and OAC is σ and the angle between the faces OAB and OBC is ρ.
Show that $\gamma > \beta \cos \sigma + \alpha \cos \rho$.

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5 Find all functions $\mathbb{R}^+ \to \mathbb{R}^+$ for which we have for all $x, y \in \mathbb{R}^+$ that $f(x)f(yf(x)) = f(x+y)$.

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6 Let A be a real $n \times n$ Matrix and define $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$
Prove or disprove that for any real polynomial $P(x)$ and any real matrices A, B, $P(e^{AB})$ is nilpotent if and only if $P(e^{BA})$ is nilpotent.
