IMC 2001

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–	Day 1
1	Let n be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, ..., n^2$ written in order, starting at the top left and moving along each row in turn left-to-right. (e.g. for $n = 3$ we get $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$) We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?
2	Let r, s, t positive integers which are relatively prime and $a, b \in G$, G a commutative multiplicative group with unit element e, and $a^r = b^s = (ab)^t = e$. (a) Prove that $a = b = e$. (b) Does the same hold for a non-commutative group G?
3	Find $\lim_{t \to 1^-} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n}$.
4	$p(x)$ is a polynomial of degree n with every coefficient 0 or ± 1, and $p(x)$ is divisible by $(x-1)^k$ for some integer $k > 0$. q is a prime such that $\frac{q}{\ln q} < \frac{k}{\ln n+1}$. Show that the complex q-th roots of unity must be roots of $p(x)$.
5	Let A be an $n \times n$ complex matrix such that $A \neq \lambda I_n$ for all $\lambda \in \mathbb{C}$. Prove that A is similar to a matrix having at most one non-zero entry on the maindiagonal.
6	Suppose that the differentiable functions $a, b, f, g : \mathbb{R} \to \mathbb{R}$ satisfy $$f(x) \geq 0, f'(x) \geq 0, g(x) \geq 0, g'(x) \geq 0 \text{ for all } x \in \mathbb{R},$$ $$\lim_{x \to \infty} a(x) = A \geq 0, \lim_{x \to \infty} b(x) = B \geq 0, \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty,$$ and $$\frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} = b(x).$$ Prove that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{B}{A+1}$.
–	Day 2

1 Let $r, s \geq 1$ be integers and $a_0, a_1, ..., a_{r-1}, b_0, b_1, ..., b_{s-1}$ be real non-negative numbers such that $(a_0 + a_1x + a_2x^2 + ... + a_{r-1}x^{r-1} + x^r)(b_0 + b_1x + b_2x^2 + ... + b_{s-1}x^{s-1} + x^s) = 1 + x + x^2 + ... + x^{r+s-1} + x^{r+s}$.
Prove that each a_i and each b_j equals either 0 or 1.

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2 Let $a_0 = \sqrt{2}, b_0 = 2, a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}, b_{n+1} = \frac{2b_n}{2+\sqrt{4+b_n^2}}$.
a) Prove that the sequences (a_n) and (b_n) are decreasing and converge to 0.
b) Prove that the sequence $(2^n a_n)$ is increasing, the sequence $(2^n b_n)$ is decreasing and both converge to the same limit.
c) Prove that there exists a positive constant C such that for all n the following inequality holds: $0 < b_n - a_n < \frac{C}{8^n}$.

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3 Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two of these points is strictly greater than $\sqrt{2}$.

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4 Let $A = (a_{k,l})_{k,l=1,...,n}$ be a complex $n \times n$ matrix such that for each $m \in \{1, 2, ..., n\}$ and $1 \leq j_1 < ... < j_m$ the determinant of the matrix $(a_{j_k,j_l})_{k,l=1,...,n}$ is zero. Prove that $A^n = 0$ and that there exists a permutation $\sigma \in S_n$ such that the matrix $(a_{\sigma(k),\sigma(l)})_{k,l=1,...,n}$ has all of its nonzero elements above the diagonal.

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5 Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$, and such that

$$f(x + y) \geq f(x) + yf(f(x)) \text{ for all } x, y \in \mathbb{R}.$$

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6 For each positive integer n, let $f_n(\vartheta) = \sin(\vartheta) \cdot \sin(2\vartheta) \cdot \sin(4\vartheta) \cdots \sin(2^n\vartheta)$.
For each real ϑ and all n, prove that

$$|f_n(\vartheta)| \leq \frac{2}{\sqrt{3}}|f_n(\frac{\pi}{3})|$$
