IMC 2002

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Day 1

1 A standard parabola is the graph of a quadratic polynomial $y = x^2 + ax + b$ with leading coëfficient 1. Three standard parabolas with vertices $V1, V2, V3$ intersect pairwise at points $A1, A2, A3$. Let $A \mapsto s(A)$ be the reflection of the plane with respect to the x-axis.

Prove that standard parabolas with vertices $s(A1), s(A2), s(A3)$ intersect pairwise at the points $s(V1), s(V2), s(V3)$.

2 Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

3 Let n be a positive integer and let $a_k = \dfrac{1}{\binom{n}{k}}, b_k = 2^{k-n},\ (k = 1..n)$.

Show that $\sum_{k=1}^{n} \dfrac{a_k - b_k}{k} = 0$.

4 Let $f : [a, b] \to [a, b]$ be a continuous function and let $p \in [a, b]$. Define $p_0 = p$ and $p_{n+1} = f(p_n)$ for $n = 0, 1, 2, ...$ Suppose that the set $T_p = \{p_n : n = 0, 1, 2, ...\}$ is closed, i.e., if $x \notin T_p$ then $\exists \delta > 0$ such that for all $x' \in T_p$ we have $|x' - x| \geq \delta$.

Show that T_p has finitely many elements.

5 Prove or disprove the following statements:
(a) There exists a monotone function $f : [0, 1] \to [0, 1]$ such that for each $y \in [0, 1]$ the equation $f(x) = y$ has uncountably many solutions x.
(b) There exists a continuously differentiable function $f : [0, 1] \to [0, 1]$ such that for each $y \in [0, 1]$ the equation $f(x) = y$ has uncountably many solutions x.

6 For an $n \times n$ matrix with real entries let $||M|| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{||Mx||_2}{||x||_2}$, where $|| \cdot ||_2$ denotes the Euclidean norm on $\mathbb{R}^n$. Assume that an $n \times n$ matrxi A with real entries satisfies $||A^k - A^{k-1}|| \leq \frac{1}{2002k}$ for all positive integers k. Prove that $||A^k|| \leq 2002$ for all positive integers k.

Day 2

7 Compute the determinant of the $n \times n$ matrix $A = (a_{ij})_{ij}$,

$$a_{ij} = \begin{cases} (-1)^{|i-j|} & \text{if } i \neq j, \\ 2 & \text{if } i = j. \end{cases}$$

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8 200 students participated in a math contest. They had 6 problems to solve. Each problem was correctly solved by at least 120 participants. Prove that there must be 2 participants such that every problem was solved by at least one of these two students.

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9 For each $n \geq 1$ let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that $a_n \cdot b_n$ is an integer.

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10 Let $OABC$ be a tetrahedon with $\angle BOC = \alpha, \angle COA = \beta$ and $\angle AOB = \gamma$. The angle between the faces OAB and OAC is σ and the angle between the faces OAB and OBC is ρ.
Show that $\gamma > \beta \cos \sigma + \alpha \cos \rho$.

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11 Let A be a complex $n \times n$ Matrix for $n > 1$. Let A^H be the conjugate transpose of A.
Prove that $A \cdot A^H = I_n$ if and only if $A = S \cdot (S^H)^{-1}$ for some complex Matrix S.

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12 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function whose gradient ∇f
exists at every point of $\mathbb{R}^n$ and satisfies the condition

$$\exists L > 0 \; \forall x_1, x_2 \in \mathbb{R}^n : \quad ||\nabla f(x_1) - \nabla f(x_2)|| \leq L||x_1 - x_2||.$$

Prove that

$$\forall x_1, x_2 \in \mathbb{R}^n : \quad ||\nabla f(x_1) - \nabla f(x_2)||^2 \leq L \langle \nabla f(x_1) - \nabla f(x_2), x_1 - x_2 \rangle.$$
